Perverse Sheaves and Finite Dimensional Algebras

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Abstract

This thesis is devoted to the study of the category of p-perverse sheaves on a topologically stratified space. In particular, we are interested in topological conditions on the space X which allow us to realise such category as finitely generated modules over a finite dimensional algebra, or equivalently as representations of a finite quiver with relations. Our approach is based on considering complementary open and closed unions of strata of X and using induction on the number of strata. The six functor formalism descends from the constructible derived category of X, the ambient triangulated category we work in, to the abelian category of p-perverse sheaves, which arises as the heart of a t-structure on the constructible derived category.

It turns out that, if X is a topologically stratified spaces with finitely many strata, each with finite fundamental group, the category of p-perverse sheaves has finitely many (isomorphism classes of) simple objects. When $\Bbbk$ is an algebraically closed field with characteristic that does not divide the order of the fundamental groups of strata, we provide a construction of projective covers (and dually injective hulls) of simple objects. This implies that the category of p-perverse sheaves has enough injectives and projectives. The (finite) sum of the projective covers of simple perverse sheaves is then a projective generator of the category of p-perverse sheaves on X. The endomorphism ring of the projective generator is a finite dimensional algebra and each p-perverse sheaf is a finitely generated module over such algebra. Moreover, one can describe the category of p-perverse sheaves as representations of the Ext-quiver with relations. We can topologically characterise the quiver and determine the quadratic part of the relations when one inductively adds one closed stratum at a time. The information needed to do so is encoded in some intersection cohomology groups of links, while in order to determine the ideal of relations one needs an A_∞-structure.

Finally, we study the representation theory of the algebra that arises as above. We give a characterisation of indecomposable p-perverse sheaves which are extensions of a given perverse sheaf over a closed stratum. As an application, we are able to determine the Auslander-Reiten quiver of the category of p-perverse sheaves on $X = \mathbb{P}^2$ with respect to the affine stratification for any GM-perversity. We can generalise some of the results for the zero and middle perversity to $X = \mathbb{P}^n$ and we give a conjecture for the Auslander-Reiten quiver of ${}^m\mathbf{Perv}(\mathbb{P}^n)$.

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To the memory of my dad.

Contents

Chapter 1

Introduction

In this PhD thesis we study the abelian category of p-perverse sheaves on a topologically stratified space X. We are particularly interested in topological conditions on the space X under which there is an equivalence between the category of p-perverse sheaves and finitely generated modules over a finitely generated algebra or, equivalently, representations of a finite quiver with relations.

Topologically stratified spaces, in the sense of [GM80, GM83], are the class of spaces we are interested in. These spaces are paracompact Hausdorff and have a filtration by closed subsets where each successive difference in the filtration is a topological manifold. The connected components of each successive difference in the filtration are called strata and usually denoted by S_i. Moreover, each point admits a neighbourhood of the form $\mathbb{R}^i \times c(L)$, where the link L is a compact lower dimensional topologically stratified space and $c(L)$ denotes the open cone $L \times [0,1)/L \times \{0\}$. One can think of a topologically stratified space as a space which globally might not be a manifold, but that can be broken into pieces, the strata, which are manifolds. These spaces appear in many different areas of mathematics such as singularity theory, algebraic geometry, differential geometry and theoretical physics. For instance, any manifold is a topologically stratified space with the trivial stratification. CW-complexes stratified by skeleta and triangulated space stratified by simplices are other examples of topologically stratified spaces. Quasi-projective varieties are topologically stratified spaces with filtration induced by their singularities, see [Whi65]. There are various refinements of the notion of stratified space where one requires extra smooth, analytic or algebraic structure. For example, a compact manifold with a Morse function gives rise to a Whitney stratified space, see [Nic11], which is the standard notion of 'smooth' topologically stratified space. We use the case of $\mathbb{P}^1$ stratified by a point and the complement as a guiding example and we try to generalise it to the case of $\mathbb{P}^n$ with the stratification given by projecting the flag of its linear subspaces. These are Whitney stratified spaces arising from the standard Morse function on $\mathbb{P}^n$.

Given a topologically stratified space X one can consider a perversity, usually denoted by p, that is a function from the set of strata of X to $\mathbb{Z}$. Such function associates a number $p(S_i) \in \mathbb{Z}$ to each stratum S_i. In this thesis, we do not make any restriction on the perversity, but the most important perversities are the so called Goresky-MacPherson ones (GM for short). These perversities have a relation between their value on strata and the geometry of the considered space. These perversities go from the zero perversity, defined to be zero on each stratum, to the top perversity, defined in terms of the codimension of each stratum, in a controlled way. Historically, particular attention has been given to the middle perversity m. This is due to the fact that such perversity plays a prominent role when one considers an algebraic variety.

Topologically stratified spaces satisfy a generalised version of the Poincaré duality for intersection (co)homology, see [GM80, GM83]. Intersection cohomology complexes turn out to be simple objects in the abelian category of perverse sheaves on a topologically stratified space. Perverse sheaves were introduced by Joseph Bernstein, Alexander Beilinson, and Pierre Deligne in the monograph [BBD82] in the early '80s. They showed that one can associate to a topologically stratified space X with a perversity p on it the category ${}^p\mathbf{Perv}(X)$ of p-perverse sheaves on X with coefficients in a field $\Bbbk$. Indeed, the category ${}^p\mathbf{Perv}(X)$ arises as the heart of a perverse t-structure on the constructible derived category $\mathbf{D}_{\mathrm{c}}(X)$, hence ${}^p\mathbf{Perv}(X)$ is an abelian subcategory of the constructible derived category. The constructible derived category is the ambient category we work in. Its objects are complexes of sheaves of $\Bbbk$-vector spaces which are cohomologically constructible, that is the restriction to each stratum of the cohomology sheaves are local systems with finitely generated fibres. The constructible derived category contains the cohomological information about all strata and links, has the six functor formalism and Verdier duality. On the other hand, $\mathbf{D}_{\mathrm{c}}(X)$ is no longer an abelian category but it is triangulated. This leads to the study of certain abelian subcategories of $\mathbf{D}_{\mathrm{c}}(X)$ which inherit the six functor formalism and behave well under Verdier duality. The theory of t-structures and hearts introduced in [BBD82] provides such setting. Indeed, one can think of the abelian category of ${}^p\mathbf{Perv}(X)$ as the result of glueing local systems on each stratum with an appropriate shift prescribed by the value of the perversity. Moreover, if one considers the zero perversity one recovers the abelian category of constructible sheaves as the heart of the standard t-structure on $\mathbf{D}_{\mathrm{c}}(X)$. Verdier duality restricts from the constructible derived category to ${}^p\mathbf{Perv}(X)$ in a way that it sends p-perverse sheaves to p^*-perverse sheaves, where p^* denotes the dual perversity on X given by $p^*(S) = -\dim(S) - p(S)$.

Our approach uses the fact that if one considers a closed union of strata Z of X and its open complement U, at the level of p-perverse sheaves the closed extension by zero i_* and open restriction j^* are exact functors which admit left and right adjoint as follows

$$\begin{tikzcd}
{}^p\mathbf{Perv}(Z) \arrow[r, "i_*", "\perp"'] & {}^p\mathbf{Perv}(X) \arrow[l, bend right, "\overset{{}^p i^*}{\perp}"'] \arrow[l, bend left, "{}^p i^!"] \arrow[r, "j^*", "\perp"'] & {}^p\mathbf{Perv}(U) \arrow[l, bend right, "\overset{{}^p j_!}{\perp}"'] \arrow[l, bend left, "{}^p j_*"]
\end{tikzcd} .$$

Furthermore, one can define the intermediate extension functor ${}^p j_{!*} : {}^p\mathbf{Perv}(U) \to {}^p\mathbf{Perv}(X)$ as the image of the natural morphism ${}^p j_! \to {}^p j_*$. The intermediate extension functor preserves monomorphisms and epimorphisms, but in general it is not exact. Such functor is crucial in order to characterise the topological information encoded in ${}^p\mathbf{Perv}(X)$. Indeed, one has that irreducible local systems on strata, that is simple objects in the category of local systems, correspond to simple perverse sheaves, which can be either a closed extension by zero of a simple perverse sheaf on Z or the intermediate extension of a simple perverse sheaf on U. Extensions between simple objects in ${}^p\mathbf{Perv}(X)$ can be described in terms of intersection cohomology groups of links.

The abelian category ${}^p\mathbf{Perv}(X)$ has some very convenient properties, see [BBD82]. It is a Krull-Remak-Schmidt category, that is every object is a direct sum of indecomposable objects. Hence, one goal is to understand its Auslander-Reiten quiver, which has (isomorphism classes of) indecomposable objects as vertices and irreducible morphisms as arrows. The category ${}^p\mathbf{Perv}(X)$ is artinian, noetherian and finite length, therefore for instance each object has a well-defined composition series. Finally, if X has finitely many strata, each with finite fundamental group, ${}^p\mathbf{Perv}(X)$ has finitely many simple objects. Throughout this thesis, we will mainly work under the latter hypothesis.

Perverse sheaves appear in the literature in different contexts. In [Pol97], the author deals with the case of triangulated spaces and proves that, in such case, the category of perverse sheaves is equivalent to finite dimensional representations of a Koszul algebra. In [Vyb98], the author studies the case of simplices. They show that there is a Koszul duality between perverse sheaves constructible with respect to a stratification and the category of sheaves constant along perverse simplices on a finite simplicial complex. In [Vyb06], the author investigates the quiver description of perverse sheaves with particular attention to the case of flag varieties, which applies to the BGG category $\mathcal{O}$. In [KS14], the authors study the case of the complexification of a real hyperplane arrangement. Although in this case the fundamental group of strata can be infinite, they show that the category of perverse sheaves admits a description as representations of a quiver with relations.

For some particular spaces or perversities, the category of perverse sheaves is equivalent to other well-known categories. For example, if one consider as a space a point, then for any perversity one recovers the category of finite $\Bbbk$-vector spaces, that is there is an equivalence of categories ${}^p\mathbf{Perv}(\{\mathrm{pt}\}) \simeq \mathbf{Vect}_\Bbbk$. A more interesting case is given by considering an unstratified manifold X. In this instance, the category of p perverse sheaves on X is equivalent to local systems on X (with a shift given by the perversity), to the category of representations of the fundamental group of X and to modules over the algebra $\Bbbk[\pi_1(X)]$. Therefore, if $\pi_1(X)$ is finite and the characteristic of the field does not divide the order of the fundamental group of X, ${}^p\mathbf{Perv}(X)$ is semisimple by Maschke's Theorem. Moreover, in this case ${}^p\mathbf{Perv}(X)$ admits a projective generator. This last situation, can be regarded as the one we want to generalise and use as base of the induction by working inductively on the number of strata. In fact, in Chapter 3 we prove the following result.

Theorem A. *Let X be a topologically stratified space and $\Bbbk$ an algebraically closed field. Then, the following conditions are equivalent:*

i) X has finitely many strata, each with finite fundamental group and the characteristic of $\Bbbk$ does not divide the order of the fundamental group of the strata.

ii) ${}^p\mathbf{Perv}(X)$ has enough injectives and projectives.

iii) There is an equivalence of categories ${}^p\mathbf{Perv}(X) \simeq A_p$-$\mathbf{mod}$, where A_p is a finite dimensional algebra.

iv) There is an equivalence of categories ${}^p\mathbf{Perv}(X) \cong \Bbbk\mathrm{Q}_p(X)/\mathrm{I}_p(X)$, where $(\mathrm{Q}_p(X), \mathrm{I}_p(X))$ is a quiver with relations.

In order to show that *i)* implies *ii)*, we use the crucial fact that, if *i)* holds, then ${}^p\mathbf{Perv}(X)$ has finitely many simple objects. We then work inductively on the number of strata and show that every simple object $\mathcal{S}_i \in {}^p\mathbf{Perv}(X)$ has a projective cover $\mathcal{P}_i$ and an injective hull $\mathcal{I}_i$. In fact, we give an explicit procedure which allows one, at least in principle, to construct projective covers and injective hulls. On the other hand, if ${}^p\mathbf{Perv}(X)$ has enough projectives (and injectives), we study under which conditions perverse functors preserve projective covers and reduce to claim about local systems. The equivalence between *ii*) and *iii*) follows from the fact that the (finite) sum of (isomorphism classes of) projective covers of simple objects $\mathcal{P} = \oplus_i^n \mathcal{P}_i$ is a projective generator for the category ${}^p\mathbf{Perv}(X)$. Hence, the functor $\mathrm{Hom}_{{}^p\mathbf{Perv}(X)}(\mathcal{P}, -) : {}^p\mathbf{Perv}(X) \to A_p\textbf{-mod}$ is an equivalence of categories where $\mathrm{A}_p = \mathrm{End}(\mathcal{P})^{op}$. The equivalence between *iii*) and *iv*) follows from the general theory of path algebras of a quiver with relations.

Theorem A holds for any perversity p on X, which can even be a non GM-perversity. Furthermore, one can inductively apply the procedure given for the construction of projective covers to determine minimal projective resolutions (and dually, minimal injective resolutions). Moreover, no complex structure is required on X. Indeed, we only used topological information about the strata of X. One needs topological information on links of strata in order to determine the algebra A_p. Theorem A extends a result in [BGS96], where the equivalences between $i), ii)$ and $iii)$ are proven for an algebraic variety with the middle perversity. We give some vanishing results for higher Ext-groups and we use these calculations to give a bound on the global dimension of ${}^p\mathbf{Perv}(X)$, under the hypothesis of faithfulness. Therefore, if ${}^p\mathbf{Perv}(X)$ is faithful in $\mathbf{D}_\mathrm{c}(X)$ the previous result is enough to guarantee the existence of a Serre functor.

We then focus on the quiver with relations $(\mathrm{Q}_p(X), \mathrm{I}_p(X))$ which appears in Theorem A. On one hand $\mathrm{Q}_p(X)$ can be completely characterised in terms of topological information. Its vertices are isomorphism classes of simple objects, while its arrows are given by Ext^1-groups between simple objects. Using the characterisation of simple objects as complexes of sheaves one can express such information in terms of intersection cohomology group of links. On the other hand, the structure of the constructible derived category is not enough in order to determine the ideal of relations $\mathrm{I}_p(X)$. In fact, the information needed to characterise $\mathrm{I}_p(X)$ is encoded in an A_∞-structure on the endomorphism algebra of the projective generator. Nevertheless, we can give a better understanding of the quadratic part of the relations $\bar{\mathrm{I}}_p(X) \subset \mathrm{I}_p(X)$ using data contained in the structure of the ambient triangulated category $\mathbf{D}_\mathrm{c}(X)$. By working inductively and adding one closed stratum at a time, we can determine the quadratic part of relations $\bar{\mathrm{I}}_p(X)$. Again, the information needed is determined by some intersection cohomology groups of links. For quadratic algebras, for example Koszul algebras, this completely determines the ideal of relations, but a complete understanding of the ideal of relations relies on being able to study the A_∞-structure.

As a consequence of Theorem A, we have that one can associate to a topologically stratified space X a family of finite dimensional algebras A_p indexed by the perversity. Therefore, we focus on the representation theory of the algebras A_p. In order to do so, we consider a closed stratum S and its open complement U in X. We then fix a perverse sheaf $\mathcal{F} \in {}^p\mathbf{Perv}(U)$ and study its extensions $\mathcal{E} \in {}^p\mathbf{Perv}(X)$, with particular attention to indecomposable ones. One can define the intermediate restriction functor ${}^pi^{!*} : {}^p\mathbf{Perv}(X) \to {}^p\mathbf{Perv}(S)$ in the same way that the intermediate extension functor has been introduced. That is, the functor ${}^pi^{!*}$ is defined as the image of the natural morphism ${}^pi^! \to {}^pi^*$. An object $\mathcal{E} \in {}^p\mathbf{Perv}(X)$ such that ${}^pi^{!*}\mathcal{E} \cong 0$ is called small. In the setting of Theorem A being small is equivalent to the fact that $\mathcal{E}$ has no summands supported on the stratum S. We then introduce two projection functors $P_!, P_* : {}^p\mathbf{Perv}(X) \to {}^p\mathbf{Perv}(X)$ which characterise extensions of $\mathcal{F}$ over S with no quotients and no sub-objects on S respectively. There is also a natural morphism $P_! \to P_*$. We introduce extension pairs of $\mathcal{F} \in {}^p\mathbf{Perv}(U)$, that is pairs of extensions of $\mathcal{F}$ which satisfy some conditions on their quotients and sub-objects supported on S. We then prove the following result under the (equivalent) conditions of Theorem A.

Theorem B. *Let $\mathcal{F} \in {}^p\mathbf{Perv}(U)$ be a fixed perverse sheaf. There is a bijection*

$$\left\{ \begin{smallmatrix} \textit{Small Extensions } \mathcal{E} \textit{ of} \\ \mathcal{F} \in {}^p\mathbf{Perv}(U) \end{smallmatrix} \right\}_{/\cong} \longleftrightarrow \left\{ \begin{smallmatrix} \textit{Extension Pairs } (\mathcal{A},\mathcal{B}) \\ \textit{relative to } \mathcal{F} \in {}^p\mathbf{Perv}(U) \end{smallmatrix} \right\}_{/\cong}.$$

Theorem B implies that we can use the projection functors $P_!, P_*$ to get 'coordinates' on the indecomposable extensions of $\mathcal{F} \in {}^p\mathbf{Perv}(U)$ over the closed stratum S. Moreover, this allows one to define the maximal extension which is the indecomposable extension of $\mathcal{F}$ in correspondence with the extension pair $({}^pj_!\mathcal{F}, {}^pj_*\mathcal{F})$. In some cases, for example complex algebraic varieties, this extension agrees with Beilinson's maximal extension, see [Bei87a, Rei10]. We then give a criterion for the indecomposability of an extension $\mathcal{E}$ of $\mathcal{F} \in {}^p\mathbf{Perv}(U)$ over a closed stratum S as follows.

Theorem C. *The extension $\mathcal{E}$ of $\mathcal{F} \in {}^p\mathbf{Perv}(U)$ over the closed stratum S is indecomposable if and only if $\mathcal{E}$ is small and the map $P_!\mathcal{E} \to P_*\mathcal{E}$ does not split.*

As a consequence of Theorems B and C, we have that in order to study the indecomposable extensions of a fixed $\mathcal{F} \in {}^p\mathbf{Perv}(U)$ over the closed stratum S, it is enough to understand pairs of objects $(P_!\mathcal{E}, P_*\mathcal{E})$ such that the map $P_!\mathcal{E} \to P_*\mathcal{E}$ does not split. One can then vary the perversity and use this approach to study all indecomposable extensions over a closed stratum for any perversity p on X. In fact, one can consider nearby perversities, that is perversities obtained by varying the value on a closed union of strata by one and study the relations between the new perverse hearts and the old one. This approach is equivalent to considering a heart which is given by tilting at a certain torsion pair. In some particular cases, for instance under the hypothesis that the starting heart is faithful, we can characterise how tilting changes simple, injective and projective objects.

We then focus on some special cases. For instance, the category of p-perverse sheaves on $X = \mathbb{P}^1$ stratified by a point and the open complement, that is with the affine stratification, is known for all the meaningful GM-perversities, that is zero, middle and top perversity, see [Woo09]. We study p-perverse sheaves on $\mathbb{P}^2$ with the affine stratification and it turns out that we are able to completely describe ${}^p\mathbf{Perv}(\mathbb{P}^2)$ for all the seven GM-perversities on $\mathbb{P}^2$. For each case, we express simple objects as complexes of sheaves and use such characterisation to determine the Ext-algebra. From this information we are able to determine the Ext-quiver $\mathrm{Q}_p(\mathbb{P}^2)$ and the relations $\mathrm{I}_p(\mathbb{P}^2)$. We then list all the indecomposable perverse sheaves, as they are the indecomposable representations of the Ext-quiver with relations, and determine the minimal projective resolutions for simple objects. We then organise this data in the Auslander-Reiten quiver of the category ${}^p\mathbf{Perv}(X)$ and find out if the considered heart is faithful or not using some ad hoc techniques. We note that ${}^m\mathbf{Perv}(\mathbb{P}^2)$ is the only faithful heart and that in one case we have an example of non-quadratic Ext-algebra. Finally, working inductively on the dimension, we consider the case $X = \mathbb{P}^n$ with the affine stratification, with particular attention to the case of the middle and zero perversity. We note that for the zero perversity, one recovers the Auslander-Reiten quiver of the path algebra over the Dynkin quiver $\mathbb{A}_{n+1}$. For the middle perversity, we are able to generalise some of the work done for $n = 2$. In particular, we propose a conjecture for the number of indecomposable objects in ${}^m\mathbf{Perv}(\mathbb{P}^n)$ and calculate the global dimension. Moreover, we give a conjectural picture of the Auslander-Reiten quiver of ${}^m\mathbf{Perv}(\mathbb{P}^n)$.

We end by posing some questions which might be interesting to investigate in future research.

1.1 Thesis Outline

In Chapter 2 we introduce the background material needed in the rest of this work. We start by recalling facts about abelian categories. Indeed, many categories we are interested in, such as finitely generated modules, representations of quivers and perverse sheaves are abelian. We then recall some facts about triangulated categories, since the ambient category we work in is the constructible derived category. Moreover, we point out some facts about t-structures, which are used to define the category of p-perverse sheaves, and tilting. We then focus on the algebraic side by recalling facts on finite dimensional algebras and quivers with relations. Particular attention is given to Auslander-Reiten theory, as one of our goals is to determine the Auslander-Reiten quiver of some categories of modules. Finally, we present the topological setting we will use. We introduce the class of spaces we will study, namely topologically stratified spaces, and the category of p-perverse sheaves. We explain the six functor formalism that descends from the constructible derived category to p-perverse sheaves and give some properties of the category ${}^p\mathbf{Perv}(X)$ for a topologically stratified space X. We conclude the chapter with some explicit examples.

In Chapter 3, we start analysing the situation of local systems. This is the situation we aim to generalise and use as base case when working inductively on the number of strata. We give a construction of projective covers for simple objects when X is a topologically stratified space with finitely many strata, each with finite fundamental group and $\Bbbk$ is an algebraic closed field with characteristic not dividing the order of the fundamental group of strata. This result leads to Theorem A. We then explain some of the consequences of such result. For example, one can inductively determine minimal projective (and injective) resolutions and use this information to calculate the Auslander-Reiten translation. Finally, we show that ${}^p\mathbf{Perv}(X)$ has finite global dimension if it is a faithful heart.

In Chapter 4, we study the quiver description of the category of perverse sheaves as representations of the Ext-quiver with relations. While the quiver can be characterised in purely topological terms, one can only describe the quadratic part of the relations by working in the constructible derived category. Indeed, the information needed to determine the whole ideal of relations is encoded in an A_∞-structure on the Ext-algebra. We show that, if one adds one closed stratum at a time and works inductively, the information required to determine the quadratic part of relations is determined by some intersection cohomology groups of links.

In Chapter 5 we study indecomposable extensions over a closed stratum, that is perverse sheaves which are extensions of a fixed perverse sheaf supported on the open complement of S. That is, we focus our attention on the Auslander-Reiten quiver of the category ${}^{p}\mathbf{Perv}(X)$. We give a criterion which allows us to identify indecomposable perverse sheaves. Finally, we study nearby perversities, that is we are interested in the relation between a fixed heart and its tilt at a torsion pair.

Chapter 6 is devoted to the study of some special cases. More particularly, we are able to fully characterise the categories ${}^{p}\mathbf{Perv}(\mathbb{P}^2)$ for all the meaningful GM-perversities, where $\mathbb{P}^2$ has the affine stratification. We then generalise some results to the case of $\mathbb{P}^n$ with the affine stratification with particular attention to the zero and middle perversity. We conclude the chapter with some final remarks and some open questions.

Chapter 2

Background

This chapter is devoted to introducing all the background material which is used in the rest of the thesis.

In Section 2.1, we present some tools from category theory employed in the other chapters. We begin by defining abelian and triangulated categories and we explain some of their features. We then briefly describe t-structures on triangulated categories, which give a way to study abelian categories inside a triangulated category. Moreover, we explain how torsion theories, a concept closely related to t-structures, allow us to study different abelian subcategories of a triangulated one by the procedure of tilting.

In Section 2.2, we deal with the algebraic concepts used later on. We recall the definition and some important facts about finite dimensional algebras, quivers and their representations. We then give a short introduction to Auslander-Reiten theory of a class of algebras which contains finite dimensional algebras. More particularly, such theory aims to study the representation theory of certain algebras, describing indecomposable objects and irreducible morphisms between them.

Finally, in Section 2.3 we present the topological notions needed in the following chapters. We introduce topologically stratified spaces, the class of spaces we are interested in, we give some examples and we define the constructible derived category of such spaces. We then give the definition of perversity function and introduce the abelian category of perverse sheaves on a topologically stratified space as the heart of the perverse t-structure on the constructible derived category. We discuss important functors at the level of perverse sheaves together with some important properties. We end the section with some special cases and well-known examples of perverse sheaves.

2.1 Categories

In this Section we recall some definitions from category theory. In particular, we will briefly introduce additive, abelian and triangulated categories together with some of their properties. The main references for this section are [HJR10] and [Lei14].

2.1.1 Abelian Categories

We introduce abelian categories and present some important features. Abelian categories are a very important tool for our purposes. In particular, the category of perverse sheaves will turn out to be abelian. Moreover, the category of finitely generated modules over an algebra and the category of representations of a quiver with relations are other two examples of abelian categories which will play a very important role in this thesis.

Definition 2.1.1.1. *A category $\mathcal{A}$ is* **additive** *if the following conditions hold:*

A1) For any pair of objects $A, B \in \mathcal{A}$ the set $\mathrm{Hom}_{\mathcal{A}}(A, B)$ is an abelian group and the composition

$$\mathrm{Hom}_{\mathcal{A}}(A, B) \times \mathrm{Hom}_{\mathcal{A}}(B, C) \to \mathrm{Hom}_{\mathcal{A}}(A, C)$$

is bilinear.

A2) There exists a zero object $0 \in \mathcal{A}$ which is both initial and final.

A3) For any pair of objects $A, B \in \mathcal{A}$ there exists a coproduct $A \oplus B \in \mathcal{A}$.

Definition 2.1.1.2. *Let $\mathcal{A}$ be an additive category. An object $A \in \mathcal{A}$ is* **indecomposable** *if it cannot be expressed as a non-trivial coproduct of objects of $\mathcal{A}$, that is if $A \cong B \oplus C$ then either $B = 0$ or $C = 0$.*

Let $\mathcal{A}$ be an additive category, the **kernel** of a morphism $f \in \mathrm{Hom}_{\mathcal{A}}(A, B)$ is an object $K \in \mathcal{A}$ together with a morphism $k \in \mathrm{Hom}_{\mathcal{A}}(K, A)$ with $f \circ k = 0$ and such that for every morphism $k' \in \mathrm{Hom}_{\mathcal{A}}(K', A)$ with $f \circ k' = 0$ there exists a unique morphism

$g \in \mathrm{Hom}_{\mathcal{A}}(K', K)$ making the diagram

$$\begin{tikzcd}
 & B & \\
A \arrow[ur, "f"] & K \arrow[l, "k"'] \arrow[u, "0"] & \\
 & & K' \arrow[ul, "g"'] \arrow[ull, "k'"] \arrow[uul, "0"']
\end{tikzcd}$$

commute. Since $\ker f$ satisfies the above universal property, if it exists it is unique (up to unique isomorphism). There is a dual notion to the one of kernel: the **cokernel** of a morphism $f \in \mathrm{Hom}_{\mathcal{A}}(A, B)$ is an object $C \in \mathcal{A}$ together with a morphism $c \in \mathrm{Hom}_{\mathcal{A}}(B, C)$ with $c \circ f = 0$ and such that for every morphism $c' \in \mathrm{Hom}_{\mathcal{A}}(B, C')$ with $c' \circ f = 0$ there exists a unique morphism $g \in \mathrm{Hom}_{\mathcal{A}}(C, C')$ making the diagram

$$\begin{tikzcd}
 & B \arrow[d, "c"] \arrow[ddr, "c'"] & \\
A \arrow[ur, "f"] \arrow[r, "0"] \arrow[drr, "0"'] & C \arrow[dr, "g"] & \\
 & & C'
\end{tikzcd}$$

commute. Again, if the cokernel exists, it is unique (up to unique isomorphism). Moreover, if the morphism $k : \ker f \to A$ has a cokernel in $\mathcal{A}$, it is called the **coimage of** f, that is $\mathrm{coim} f = \mathrm{coker} k$. Dually, if the morphism $c : B \to \mathrm{coker} f$ has a kernel in $\mathcal{A}$, it is called the **image of** f, that is $\mathrm{im} f = \ker c$. Furthermore, if a morphism $f \in \mathrm{Hom}_{\mathcal{A}}(A, B)$ has both coimage and image then, by their universal properties, there is a natural morphism $\mathrm{coim} f \to \mathrm{im} f$.

Definition 2.1.1.3. *An additive category* $\mathcal{A}$ *is* **abelian** *if the following conditions are satisfied:*

A4) For any $f \in \mathrm{Hom}_{\mathcal{A}}(A, B)$ *there exist* $\ker f$ *and* $\mathrm{coker} f$*.*

A5) For any $f \in \mathrm{Hom}_{\mathcal{A}}(A, B)$ *the morphism* $\mathrm{coim} f \to \mathrm{im} f$ *is an isomorphism.*

Example 2.1.1.4. *The category* **Ab** *of abelian groups,* $\mathbf{Vect}_{\Bbbk}$ *of vector spaces over a field* $\Bbbk$ *are abelian categories. Let* $\mathcal{A}$ *be an abelian category and denote by* $\mathrm{C}(\mathcal{A})$ *the category of* *chain complexes in* $\mathcal{A}$*, that is the category with objects* $A_\bullet = (A_n, d_n)_{n\in\mathbb{Z}}$*, where* $A_n \in \mathcal{A}$ *and* $d_n \in \mathrm{Hom}_{\mathcal{A}}(A_{n+1}, A_n)$ *are such that* $d_n \circ d_{n+1} = 0$*, and morphisms given by chain maps, that is* $f \in \mathrm{Hom}_{\mathrm{C}(\mathcal{A})}(A_\bullet, B_\bullet)$ *is a collection of morphisms* $\{f_n : A_n \to B_n\}_{n\in\mathbb{Z}}$ *in* $\mathcal{A}$ *such that* $f_n \circ d_n^A = d_n^B \circ f_{n+1}$*. The category* $\mathrm{C}(\mathcal{A})$ *of complexes in an abelian category* $\mathcal{A}$ *is abelian, see [HJR10, Proposition 2.5]. The category of cochain complexes in* $\mathcal{A}$ *denoted by* $\mathrm{Co}(\mathcal{A})$*, that is of chain complexes in* $\mathcal{A}^{op}$*, is another example of an abelian category.*

Definition 2.1.1.5. *An abelian category* $\mathcal{A}$ *is* **$\Bbbk$-linear** *if the groups* $\mathrm{Hom}_{\mathcal{A}}(A, B)$ *are equipped with a structure of* $\Bbbk$*-vector space and the composition of maps is* $\Bbbk$*-linear in each argument.*

We now recall some definitions of some important classes of morphisms in an abelian category. In turn, this allows us to introduce sub-objects and quotients of a certain object.

Definition 2.1.1.6. *Let* $\mathcal{A}$ *be an abelian category. A morphism* $f \in \mathrm{Hom}_{\mathcal{A}}(A, B)$ *is* **monic***, or a* **monomorphism***, if* $\ker f = 0$*. We use the notation* $f : A \hookrightarrow B$*. Dually,* $f \in \mathrm{Hom}_{\mathcal{A}}(A, B)$ *is* **epic***, or an* **epimorphism***, if* $\mathrm{coker} f = 0$ *and in such case we denote it by* $f : A \twoheadrightarrow B$*. Finally,* $f \in \mathrm{Hom}_{\mathcal{A}}(A, B)$ *is an* **isomorphism** *if it is both monic and epic. In this latter case we write* $A \cong B$*.*

Definition 2.1.1.7. *Let* $\mathcal{A}$ *be an abelian category. Two monomorphisms* $f \in \mathrm{Hom}_{\mathcal{A}}(A, B)$ *and* $f' \in \mathrm{Hom}_{\mathcal{A}}(A', B)$ *are* **equivalent** *if there exists an isomorphism* $h \in \mathrm{Hom}_{\mathcal{A}}(A, A')$ *such that* $f = f' \circ h$*, that is if the diagram*

$$\begin{tikzcd} A \arrow[r, hook, "f"] \arrow[d, "h"'] & B \\ A' \arrow[ur, hook, "f'"'] & \end{tikzcd}$$

commutes. Note that h *is unique.*

Definition 2.1.1.8. *Let* $\mathcal{A}$ *be an abelian category. A* **sub-object** *of* $B \in \mathcal{A}$ *is an equivalence class of monomorphisms* $f \in \mathrm{Hom}_{\mathcal{A}}(A, B)$*. We refer to* A *as the sub-object of* B *and we write* $A \subset B$*. Moreover, we denote by* 0 *and* B *the sub-objects of* B *which are equivalence classes of* $0 \to B$ *and* $\mathrm{id}_B : B \to B$*. Dually, a* **quotient** *of* $B \in \mathcal{A}$ *is an equivalence class of epimorphisms* $g \in \mathrm{Hom}_{\mathcal{A}}(B, C)$*.*

Remark 2.1.1.9. *Note that one can define the operations of sum, denoted by* $\sum$*, and intersection, denoted by* $\bigcap$*, on sub-objects, see [Pop73, Section 2.6].*

Definition 2.1.1.10. *Let $\mathcal{A}$ be an abelian category. An object $A \in \mathcal{A}$ is* **simple** *if it has only* 0 *and A as sub-objects.*

Definition 2.1.1.11. *Let $\mathcal{A}$ be an abelian category. An object $A \in \mathcal{A}$ is* **semisimple** *if it is a direct sum of simple objects.*

Definition 2.1.1.12. *Let $\mathcal{A}$ be an abelian category. A* **composition series** *for $A \in \mathcal{A}$ is a finite sequence of monomorphisms*

$$0 = A_0 \hookrightarrow A_1 \hookrightarrow \ldots \hookrightarrow A_{n-1} \hookrightarrow A_n = A$$

such that the quotient A_i/A_{i-1} is a simple object $\forall i$. The integer $\ell(A) = n$ is called the **length of** *A.*

Definition 2.1.1.13. *An abelian category is* **finite length** *if every object has a composition series as in Definition 2.1.1.12.*

Remark 2.1.1.14. *The length $\ell(A)$ of an object $A \in \mathcal{A}$ in a finite length category is well-defined by the Jordan-Hölder Theorem for abelian categories, see [Ses67, Theorem 2.1].*

Definition 2.1.1.15. *Let $\mathcal{A}$ be an abelian category. We say that:*

i) *$\mathcal{A}$ is* **noetherian** *if for every $A \in \mathcal{A}$ every ascending chain of sub-objects of A stabilises.*

ii) *$\mathcal{A}$ is* **artinian** *if for every $A \in \mathcal{A}$ every descending chain of sub-objects of A stabilises.*

Definition 2.1.1.16. *Let $\mathcal{A}$ be an abelian category. An* **exact sequence** *in $\mathcal{A}$ is a sequence of objects and composable morphisms in $\mathcal{A}$ of the form*

$$\ldots \to A_{i-1} \xrightarrow{f_{i-1}} A_i \xrightarrow{f_i} A_{i+1} \to \ldots$$

such that $\mathrm{im} f_{i-1} = \ker f_i$. *A* **short exact sequence** *in $\mathcal{A}$ is an exact sequence of the form*

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$$

where f is monic, g is epic and $\mathrm{im} f \cong \ker g$.

Remark 2.1.1.17. *Let $\mathcal{A}$ be an abelian category. For any two objects $A, B \in \mathcal{A}$ and for any $n \geq 0$ we can define the abelian groups $\mathrm{Ext}^n_{\mathcal{A}}(A, B)$, called the* **(Yoneda)** Ext**-groups**, *by setting:*

i) $\mathrm{Ext}^0_{\mathcal{A}}(A, B) = \mathrm{Hom}_{\mathcal{A}}(A, B)$.

ii) $\mathrm{Ext}^1_{\mathcal{A}}(A, B)$ *to be the set of equivalence classes of extensions of* A *by* B*, that is of short exact sequences of the form* $0 \to B \to E \to A \to 0$ *in* $\mathcal{A}$*, where two extensions are equivalent if there is a commutative diagram*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & B & \longrightarrow & E & \longrightarrow & A & \longrightarrow & 0 \\ & & \| & & \downarrow & & \| & & \\ 0 & \longrightarrow & B & \longrightarrow & E' & \longrightarrow & A & \longrightarrow & 0 \end{array}.$$

The Baer sum gives $\mathrm{Ext}^1_{\mathcal{A}}(A, B)$ *the structure of abelian group, with the trivial extension, that is the one with middle term* $E \cong A \oplus B$*, which serves as the zero element, see [Mit65, Chapter VII, Theorem 1.5].*

iii) The higher Ext*-groups* $\mathrm{Ext}^n_{\mathcal{A}}(A, B)$ *are defined as equivalence classes of* n*-extensions, that is of exact sequences* $0 \to B \to X_n \to \ldots \to X_1 \to A \to 0$*, where two extensions are equivalent if there is a commutative diagram*

$$\begin{array}{ccccccccccccc} 0 & \longrightarrow & B & \longrightarrow & X_n & \longrightarrow & \ldots & \longrightarrow & X_1 & \longrightarrow & A & \longrightarrow & 0 \\ & & \| & & \downarrow & & & & \downarrow & & \| & & \\ 0 & \longrightarrow & B & \longrightarrow & X'_n & \longrightarrow & \ldots & \longrightarrow & X'_1 & \longrightarrow & A & \longrightarrow & 0 \end{array}.$$

Again, the Baer sum gives $\mathrm{Ext}^n_{\mathcal{A}}(A, B)$ *the structure of abelian group.*

2.1.2 Simple, Projective and Injective Objects

Some particular objects, namely simple, projective and dually injective objects, play a central role in the context of abelian categories.

Definition 2.1.2.1. *Let* $\mathcal{A}$ *be an abelian category. An object* $P \in \mathcal{A}$ *is* **projective** *if one of the following equivalent conditions hold:*

i) $\mathrm{Hom}_{\mathcal{A}}(P, -)$ *is exact.*

ii) $\mathrm{Ext}^1_{\mathcal{A}}(P, -) = 0$.

iii) For any epimorphism $\varphi \in \mathrm{Hom}_{\mathcal{A}}(A, B)$ *and morphism* $f \in \mathrm{Hom}_{\mathcal{A}}(P, B)$ *there exists* *a morphism* $f' \in \mathrm{Hom}_{\mathcal{A}}(P, A)$ *such that the diagram*

$$\begin{tikzcd} & P \arrow[dl, dashed, "\exists f'"'] \arrow[d, "f"] \\ A \arrow[r, "\varphi"'] & B \end{tikzcd}$$

commutes, that is $f = \varphi \circ f'$*.*

iv) Any epimorphism $A \twoheadrightarrow P$ *splits.*

We briefly sketch the equivalences between the four conditions of Definition 2.1.2.1.

ii)⇒i): Let $0 \to A \to B \to C \to 0$ be a short exact sequence in $\mathcal{A}$. Then applying the left exact functor $\mathrm{Hom}_{\mathcal{A}}(P, -)$ to it yields the long exact sequence

$$0 \to \mathrm{Hom}_{\mathcal{A}}(P, A) \to \mathrm{Hom}_{\mathcal{A}}(P, B) \to \mathrm{Hom}_{\mathcal{A}}(P, C) \to \mathrm{Ext}^1_{\mathcal{A}}(P, A) \to \dots$$

If $\mathrm{Ext}^1_{\mathcal{A}}(P, -) = 0$, then $\mathrm{Hom}_{\mathcal{A}}(P, -)$ is also right exact and hence exact.

i) $\Longleftrightarrow$ iii): If $\mathrm{Hom}_{\mathcal{A}}(P, -)$ is exact, then for any epimorphism $\varphi : A \twoheadrightarrow B$ we have $\mathrm{Hom}_{\mathcal{A}}(P, A) \twoheadrightarrow \mathrm{Hom}_{\mathcal{A}}(P, B)$. Therefore, for any epimorphism $\varphi \in \mathrm{Hom}_{\mathcal{A}}(A, B)$ and $f \in \mathrm{Hom}_{\mathcal{A}}(P, B)$ there exists $f' \in \mathrm{Hom}_{\mathcal{A}}(A, B)$ such that $\varphi \circ f' = f$. Vice versa, the lifting property implies that there is an epimorphism $\mathrm{Hom}_{\mathcal{A}}(P, A) \twoheadrightarrow \mathrm{Hom}_{\mathcal{A}}(P, B)$ for any $A \twoheadrightarrow B$ in $\mathcal{A}$, hence $\mathrm{Hom}_{\mathcal{A}}(P, -)$ is exact.

i)⇒iv): If we apply the exact functor $\mathrm{Hom}_{\mathcal{A}}(P, -)$ to an epimorphism $\alpha : A \twoheadrightarrow B$ in $\mathcal{A}$ we get $\mathrm{Hom}_{\mathcal{A}}(P, A) \twoheadrightarrow \mathrm{Hom}_{\mathcal{A}}(P, B)$. Hence, the identity $\mathrm{id}_P : P \to P$ lifts to a morphism $P \to A$ and therefore α splits.

iv)⇒ii): If any epimorphism $\alpha : A \twoheadrightarrow B$ in $\mathcal{A}$ splits, then so does any short exact sequence of the form $0 \to \ker \alpha \to A \to P \to 0$, hence $\mathrm{Ext}^1_{\mathcal{A}}(P, -) = 0$.

Definition 2.1.2.2. *An abelian category* $\mathcal{A}$ *has* **enough projectives** *if any object is a quotient of a projective object.*

Definition 2.1.2.3. *Let* $\mathcal{A}$ *be an abelian category. A* **projective cover** *of an object* $A \in \mathcal{A}$ *is a pair* (P, π) *such that:*

i) $P \in \mathcal{A}$ *is projective.*

ii) *There exists an epimorphism* $\pi \in \mathrm{Hom}_{\mathcal{A}}(P, A)$.

iii) *If the diagram*

$$\begin{tikzcd} P \arrow[rr, "\alpha"] \arrow[dr, two heads, "\pi"'] & & P \arrow[dl, two heads, "\pi"] \\ & A & \end{tikzcd}$$

commutes, then α *is an isomorphism.*

Remark 2.1.2.4. *Often we abuse terminology by referring to* P *as the projective cover leaving the map* π *in the definition of projective cover as understood. Note that if a projective cover exists it is unique up to isomorphism, see [Kra14, Corollary 3.5]. Furthermore, the projective cover of a simple object is indecomposable, see [Kra14, Lemma 3.6].*

Definition 2.1.2.5. *Let* $\mathcal{A}$ *be a* Hom*-finite Krull-Schmidt abelian category. A* **projective generator** *of* $\mathcal{A}$ *is an object* $P_{\mathcal{A}} \in \mathcal{A}$ *satisfying the following two properties:*

i) $P_{\mathcal{A}}$ *is projective.*

ii) *For any object* $S \in \mathcal{A}$ *there is an epimorphism* $P_{\mathcal{A}}^n \twoheadrightarrow S$ *for some* $n \in \mathbb{N}$.

For finite length categories with finitely many simple objects, we can give another characterisation of projective covers of simple objects.

Lemma 2.1.2.6. *Let* $\mathcal{A}$ *be a* $\Bbbk$*-linear category of finite length with finitely many simple objects* $\{S_i \mid i \in I\}$. *An object* $P \in \mathcal{A}$ *is the projective cover of* S_i *if:*

i) $\mathrm{Hom}_{\mathcal{A}}(P, S_k) \cong \begin{cases} \Bbbk & \text{if } k = i \\ 0 & \text{otherwise.} \end{cases}$

ii) $\mathrm{Ext}^1_{\mathcal{A}}(P, -) = 0$.

Proof. First note that if there is a morphism $\pi : P \to S_i$, then it is an epimorphism. Indeed, its cokernel is the quotient of S_i by a non-zero sub-object. Since S_i is simple such sub-object must be S_i, hence $\mathrm{coker}\pi = 0$. Moreover, P is indecomposable as if $P = P' \oplus P''$ without loss of generality we can assume $\mathrm{Hom}_{\mathcal{A}}(P', S_i) = 0$. Hence $\mathrm{Hom}_{\mathcal{A}}(P', S_k) = 0$ for any $k \in I$ and so $P' = 0$. Let us consider the commutative diagram

$$\begin{tikzcd} P \arrow[rr, "\alpha"] \arrow[dr, two heads, "\pi"'] & & P \arrow[dl, two heads, "\pi"] \\ & S_i & \end{tikzcd}.$$

Since P is indecomposable, by the Fitting Lemma, see [Kra14, Lemma 5.3], α is either nilpotent or invertible. If α is nilpotent, that is $\exists n$ such that $\alpha^n = 0$, we have

$$\pi = \pi\alpha^n = 0$$

which is a contradiction. Thus α is invertible. Finally condition ii) implies that P is projective, therefore P is the projective cover of S_i. □

Definition 2.1.2.7. *Let $\mathcal{A}$ be an abelian category. A* **projective resolution** *of an object $A \in \mathcal{A}$ is a chain complex*

$$P_\bullet = \ldots \to P_k \to \ldots \to P_1 \to P_0 \to 0 \to \ldots$$

with $P_i \in \mathcal{A}$ projective $\forall i$, such that there is $P_0 \to A$ for which

$$\ldots \xrightarrow{d_k} P_k \xrightarrow{d_{k-1}} P_{k-1} \xrightarrow{d_{k-2}} \ldots \to P_1 \xrightarrow{d_0} P_0 \xrightarrow{p_0} A \to 0$$

is exact. A projective resolution is **minimal** *if P_0 is the projective cover of A, P_1 is the projective cover of $\ker p_0$ and P_i is the projective cover of $\ker(d_{i-2})$ for any $i \geq 2$.*

All the above material regarding projective objects has a completely dual counterpart in terms of injective objects.

Definition 2.1.2.8. *Let $\mathcal{A}$ be an abelian category. An object $I \in \mathcal{A}$ is* **injective** *if one of the following equivalent conditions hold:*

i) $\mathrm{Hom}_{\mathcal{A}}(-, I)$ *is exact.*

ii) $\mathrm{Ext}^1_{\mathcal{A}}(-, I) = 0$.

iii) For any monomorphism $\varphi \in \mathrm{Hom}_{\mathcal{A}}(A, B)$ and map $f \in \mathrm{Hom}_{\mathcal{A}}(A, I)$ there exists $f' \in \mathrm{Hom}_{\mathcal{A}}(B, I)$ such that the diagram

$$\begin{tikzcd} A \arrow[r, hook, "\varphi"] \arrow[d, "f"'] & B \arrow[dl, dashed, "\exists f'"] \\ I & \end{tikzcd}$$

commutes, that is $f = f' \circ \varphi$.

iv) Any monomorphism $I \hookrightarrow A$ splits.

Definition 2.1.2.9. *An abelian category $\mathcal{A}$ has* **enough injectives** *if any object is a sub-object of an injective object.*

Definition 2.1.2.10. *Let $\mathcal{A}$ be an abelian category. An* **injective hull** *of an object $A \in \mathcal{A}$ is a pair (I, i) such that:*

i) $I \in \mathcal{A}$ is injective.

ii) There exists a monomorphism $i \in \mathrm{Hom}_{\mathcal{A}}(A, I)$.

iii) If the diagram

$$\begin{tikzcd} & A \arrow[dl, hook', "i"'] \arrow[dr, hook, "i"] & \\ P \arrow[rr, "\alpha"] & & P \end{tikzcd}$$

is commutative, then α is an isomorphism.

Remark 2.1.2.11. *Dually to Remark 2.1.2.4, the map in the definition of injective hull is often left as understood. Moreover, if an injective hull exists it is unique up to isomorphism and the injective hull of a simple object is indecomposable.*

Lemma 2.1.2.12. *Let $\mathcal{A}$ be a $\Bbbk$-linear category of finite length with finitely many simple objects $\{S_i \mid i \in I\}$. An object $I \in \mathcal{A}$ is the injective hull of S_i if:*

i) $\mathrm{Hom}_{\mathcal{A}}(S_k, I) \cong \begin{cases} \Bbbk & \text{if } k = i \\ 0 & \text{otherwise.} \end{cases}$

ii) $\mathrm{Ext}^1_{\mathcal{A}}(-, I) = 0$.

Proof. One can use the dual argument to the one in the proof of Lemma 2.1.2.6. □

Definition 2.1.2.13. *Let $\mathcal{A}$ be an abelian category. An* **injective resolution** *of an object $A \in \mathcal{A}$ is a cochain complex*

$$I^\bullet = \ldots \to 0 \to I^0 \to I^1 \to \ldots \to I^k \to \ldots$$

where $I^n \in \mathcal{A}$ are injective objects $\forall n$ such that there is $A \to I^0$ for which

$$0 \to A \xrightarrow{i^0} I^0 \xrightarrow{d^0} I^1 \xrightarrow{d^1} \ldots \xrightarrow{d^{k-1}} I^k \xrightarrow{d^k} I^{k+1} \xrightarrow{d^{k+1}} \ldots$$

is exact. An injective resolution is **minimal** *if* I^0 *is the injective hull of* A, I^1 *is the injective hull of* $\operatorname{coker} i^0$ *and* I^n *is the injective hull of* $\operatorname{coker}(d^{n-2})$ *for any* $n \geq 2$.

Let $\Bbbk$ be an algebraically closed field and A a finite dimensional $\Bbbk$-algebra. We denote by A-**Mod** the category of (right) modules over the algebra A. A right A-module $M \in A$-**Mod** is **finitely generated** by the elements $m_1, \dots, m_n \in M$ if any $m \in M$ can be written as $m = m_1a_1 + \dots + m_na_n$ for some $a_1, \dots, a_n \in A$. We denote by A-**mod** the full abelian subcategory of A-**Mod** consisting of finitely generated modules.
We now recall the Freyd-Mitchell embedding theorem, which provides a bridge between abelian categories and categories of modules over a ring.

Theorem 2.1.2.14 (Freyd-Mitchell)**.** *Every small abelian category $\mathcal{A}$ admits a full, faithful and exact functor*

$$\mathcal{A} \to R\text{-}\mathbf{Mod}$$

for a ring R. *In particular, $\mathcal{A}$ is equivalent to a full subcategory of R-**Mod**.*

The above result allows us to state some important definitions and results for the category of modules, the case the literature mostly deals with. In the rest of the thesis, we will only deal with the specific case of modules over a finite dimensional algebra A.

Definition 2.1.2.15. *The* **radical** *of a module $M \in A$-**Mod** is given by*

$$\operatorname{rad}(M) = \bigcap \{N \subseteq M \mid M/N \textit{ is simple}\} \subseteq M.$$

Definition 2.1.2.16. *The* **head** *(or* **top***) of a finitely generated module $M \in A$-**mod** is given by*

$$\operatorname{top}(M) = M/\operatorname{rad}(M).$$

Remark 2.1.2.17. *The head* $\operatorname{top}(M)$ *of a module $M \in A$-**mod** is semisimple, see [ASS06, I, Corollary 3.8].*

Lemma 2.1.2.18. *The projective cover of a finitely generated module $M \in A$-**mod** coincides with the projective cover of its head.*

Proof. Let us consider the diagram

$$\begin{tikzcd}
 & & & \mathcal{P}(\operatorname{top}(M)) \arrow[dl, dashed, "\alpha"'] \arrow[d, two heads] & \\
0 \arrow[r] & \operatorname{rad}(M) \arrow[r] & M \arrow[r] & \operatorname{top}(M) \arrow[r] & 0
\end{tikzcd}$$

where $\mathcal{P}(-)$ denotes the projective cover of an object. Then, $M = \mathrm{rad}(M) + \mathrm{im}\alpha$, but the fact that M is finitely generated is equivalent to $\mathrm{rad}(M)$ being superfluous, see [AF92, Theorem 10.4]. Therefore, $M \cong \mathrm{im}\alpha$ and $\mathcal{P}(M) \cong \mathcal{P}(\mathrm{top}(M))$ $\square$

The radical of a module has a dual notion, the socle of a module.

Definition 2.1.2.19. *The* **socle** *of a module $M \in A$-**Mod** is given by*

$$\mathrm{soc}(M) = \sum \{N \subseteq M \mid N \textit{ simple}\}.$$

Theorem 2.1.2.20. *There is a one-to-one correspondence (up to isomorphism) between simple modules, indecomposable projectives modules and indecomposable injectives modules. That is there are bijections*

$$\left\{\begin{matrix}\textit{indecomposable}\\ \textit{injective objects}\end{matrix}\right\} \underset{I_S \leftarrow\!\shortmid S}{\overset{I\mapsto \mathrm{soc}(I)}{\rightleftarrows}} \left\{\begin{matrix}\textit{simple}\\ \textit{objects}\end{matrix}\right\} \underset{\mathrm{top}(P)\leftarrow\!\shortmid P}{\overset{S\mapsto P_S}{\rightleftarrows}} \left\{\begin{matrix}\textit{indecomposable}\\ \textit{projective objects}\end{matrix}\right\}.$$

Proof. See [Lei15, Theorem 7.1] for the right hand side and use duality to get the left hand side. $\square$

2.1.3 Triangulated Categories

In this section we introduce triangulated categories, introduced for the first time by Jean-Louis Verdier in his thesis in 1963. Triangulated categories provide a very general and flexible setting which allows us to do homological algebra. For example, the notion of short exact sequence is replaced by triangles. In particular, triangulated categories axiomatise the structure of the derived category of an abelian category which, in general, is not abelian. The main references for this section are [HJR10, GM13, GM94].

Definition 2.1.3.1. *Let $\mathcal{T}$ be an additive category and $\Sigma : \mathcal{T} \to \mathcal{T}$ be an additive auto-equivalence. A* **triangle** *in $\mathcal{T}$ is a sequence of objects and morphisms in $\mathcal{T}$ of the form*

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X.$$

Definition 2.1.3.2. *A* **morphism of triangles** *is a commutative diagram in $\mathcal{T}$ of the* *form*

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, "f"] & Y \arrow[r, "v"] \arrow[d, "g"] & Z \arrow[r, "w"] \arrow[d, "h"] & \Sigma X \arrow[d, "\Sigma f"] \\ X' \arrow[r, "u'"] & Y' \arrow[r, "v'"] & Z' \arrow[r, "w'"] & \Sigma X' \end{tikzcd} .$$

Definition 2.1.3.3. *A* **triangulated category** *is a triple* $(\mathcal{T}, \Sigma, \mathrm{D})$ *where* $\mathcal{T}$ *is an additive category,* $\Sigma : \mathcal{T} \to \mathcal{T}$ *is an additive auto-equivalence and* D *is a class of triangles, called* **distinguished triangles**, *satisfying the following axioms:*

TR0) Any triangle isomorphic to a distinguished triangle is a distinguished triangle.

TR1) For any object $X \in \mathcal{T}$ *the triangle*

$$X \xrightarrow{\mathrm{id}} X \longrightarrow 0 \longrightarrow \Sigma X$$

is a distinguished triangle.

TR2) For every morphism $f \in \mathrm{Hom}_{\mathcal{T}}(X, Y)$ *there is a distinguished triangle of the form*

$$X \xrightarrow{f} Y \longrightarrow Z \longrightarrow \Sigma X.$$

TR3) The triangle

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X$$

is distinguished if and only if the triangle

$$Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X \xrightarrow{-\Sigma u} \Sigma Y$$

is distinguished.

TR4) Given two distinguished triangles, any commutative diagram of the form

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, "f"] & Y \arrow[r, "v"] \arrow[d, "g"] & Z \arrow[r, "w"] \arrow[d, dashed] & \Sigma X \arrow[d, "\Sigma f"] \\ X' \arrow[r, "u'"] & Y' \arrow[r, "v'"] & Z' \arrow[r, "w'"] & \Sigma X' \end{tikzcd}$$

can be competed (not necessarily uniquely) to a morphism of triangles.

TR5) Octahedral axiom: given three distinguished triangles

$$X \xrightarrow{u} Y \longrightarrow Z' \longrightarrow \Sigma X$$

$$Y \xrightarrow{v} Z \longrightarrow X' \longrightarrow \Sigma Y$$

$$X \xrightarrow{v\circ u} Z \longrightarrow Y' \longrightarrow \Sigma X$$

there exists a distinguished triangle

$$Z' \longrightarrow Y' \longrightarrow X' \longrightarrow \Sigma Z'$$

making the following diagram

$$\begin{tikzcd}
X \arrow[r, "u"] \arrow[d, "\mathrm{id}_X"] & Y \arrow[r] \arrow[d, "v"] & Z' \arrow[r] \arrow[d] & \Sigma X \arrow[d, "\mathrm{id}_{\Sigma X}"] \\
X \arrow[r, "v\circ u"] \arrow[d, "u"] & Z \arrow[r] \arrow[d, "\mathrm{id}_Z"] & Y' \arrow[r] \arrow[d] & \Sigma X \arrow[d, "\Sigma u"] \\
Y \arrow[r, "v"] \arrow[d] & Z \arrow[r] \arrow[d] & X' \arrow[r] \arrow[d, "\mathrm{id}_{X'}"] & \Sigma Y \arrow[d] \\
Z' \arrow[r] & Y' \arrow[r] & X' \arrow[r] & \Sigma Z'
\end{tikzcd}$$

commute.

Remark 2.1.3.4. *Axiom TR5) in Definition 2.1.3.3 is called octahedral axiom since in Verdier's thesis the objects and morphisms involved are organised as vertices and edges of an octahedron. In particular, four of the faces of the octahedron are distinguished triangles. In [BBD82], the same axiom is presented using the picture of two pyramids which represent the lower and upper cap of the octahedron. Moreover, if one can express the third terms in the three triangles of TR5) as* $Z' = Y/X$, $X' = Z/Y$ *and* $Y' = Z/X$ *the existence of the fourth triangle gives the Third Isomorphism Theorem as one can write* $Z/Y = (Z/X)/(Y/X)$.

Example 2.1.3.5. *Let* $\mathcal{A}$ *be an abelian category and let denote by* $\mathrm{C}^b(\mathcal{A})$ *the category of bounded chain complexes in* $\mathcal{A}$*, that is complexes* $A_\bullet = (A_n, d_n)_{n\in\mathbb{Z}}$ *such that* $A_n = 0$ *for* $|n| > k$ *for some* $k \in \mathbb{Z}$*. One can introduce the notion of homotopy between two morphisms in* $\mathrm{C}^b(\mathcal{A})$*, see [HJR10, Section 1.2]. Then, one can define the bounded homotopy category* $\mathrm{K}^b(\mathcal{A})$ *as the category having same objects as* $\mathrm{C}^b(\mathcal{A})$ *and equivalence classes of morphisms* *in* $\mathrm{C}^b(\mathcal{A})$ *modulo homotopy, see [HJR10, Definition 1.6]. While the category* $\mathrm{C}^b(\mathcal{A})$ *is abelian, see [HJR10, Proposition 2.5], the homotopy category* $\mathrm{K}^b(\mathcal{A})$ *in general is not. The failure of condition A4) of Definition 2.1.1.3 is given in [HJR10, Example 2.6] for* $\mathcal{A} = \mathbf{Ab}$*. On the other hand, the bounded homotopy category* $\mathrm{K}^b(\mathcal{A})$ *is triangulated, see [HJR10, Theorem 6.7]. Moreover, one can define the bounded derived category of* $\mathcal{A}$*, denoted by* $\mathbf{D}^b(\mathcal{A})$*, by localising* $\mathrm{K}^b(\mathcal{A})$ *at the class of quasi-isomorphisms, that is morphisms in* $\mathrm{K}^b(\mathcal{A})$ *which induce an isomorphism in homology. The bounded derived category of* $\mathcal{A}$ *turns out to be another example of triangulated category, see [HJR10, Theorem 7.18].*

The construction described in Example 2.1.3.5 works for the bounded below, bounded above and unbounded cases as well. In order to avoid some technicalities, in this thesis we will always deal with the bounded situation, therefore in the rest of this thesis we will write $\mathbf{D}(\mathcal{A})$ for the bounded derived category of $\mathcal{A}$.

Definition 2.1.3.6. *Let* $\mathcal{T}$ *be a triangulated category and* $\mathcal{A}$ *an abelian category. An additive functor* $H : \mathcal{T} \to \mathcal{A}$ *is* **cohomological** *if for any distinguished triangle (in* $\mathcal{T}$*)*

$$X \longrightarrow Y \longrightarrow Z \longrightarrow \Sigma X$$

the sequence

$$H(X) \to H(Y) \to H(Z)$$

is exact (in $\mathcal{A}$*).*

Remark 2.1.3.7. *Let* H *be a cohomological functor as in Definition 2.1.3.6, if we set*

$$H^i(X) = H(\Sigma^i X) = H \circ \Sigma^i X$$

then the sequence

$$\ldots \to H^{i-1}(Z) \to H^i(X) \to H^i(Y) \to H^i(Z) \to H^{i+1}(X) \to \ldots$$

is exact by axiom TR3).

Example 2.1.3.8. *Let* $\mathcal{T}$ *be a triangulated category. For any* $X \in \mathcal{T}$ *the functor*

$$\mathrm{Hom}_{\mathcal{T}}(X, -) : \mathcal{T} \to \mathbf{Ab}$$

is cohomological, see [GM94, Chapter 5, 1.6].

Definition 2.1.3.9. *Let $\mathcal{T}$ be a triangulated category and $X, Y \in \mathcal{T}$. The* Ext*-groups in the triangulated category $\mathcal{T}$ are defined as*

$$\mathrm{Ext}^n_{\mathcal{T}}(X, Y) = \mathrm{Hom}_{\mathcal{T}}(X, Y[n]).$$

A crucial problem is trying to relate the above Definition of Ext-groups in a triangulated category with the notion of Yoneda Ext-groups, see Remark 2.1.1.17. For now, if $\mathcal{A}$ is an abelian category and $\mathcal{T} = \mathbf{D}(\mathcal{A})$ then $\mathrm{Ext}^n_{\mathcal{T}}(X, Y) = \mathrm{Ext}^n_{\mathcal{A}}(X, Y)$ for $X, Y \in \mathcal{A} \subset \mathcal{T}$, considering the objects X and Y as complexes concentrated in degree zero. We will further discuss this in the next sections.

2.1.4 t-structures

We now present the crucial concept of t-structure on a triangulated category which allows us to study certain abelian categories inside a triangulated category. The main reference for this part is [BBD82] as well as [Dim04, KS13].
First, we recall several definitions from general category theory.

Definition 2.1.4.1. *Let $\mathcal{A}$ be a category. A subcategory $\mathcal{B} \subset \mathcal{A}$ is* **full** *if*

$$\mathrm{Hom}_{\mathcal{B}}(A, B) \cong \mathrm{Hom}_{\mathcal{A}}(A, B) \quad \forall A, B \in \mathcal{B}.$$

Definition 2.1.4.2. *A subcategory $\mathcal{B}$ of $\mathcal{A}$ is* **strictly full** *if it is full and closed under isomorphism, that is given $B \in \mathcal{B}$ whenever $B' \cong B$ then $B' \in \mathcal{B}$.*

Definition 2.1.4.3. *Let $\mathcal{B}$ be a subcategory of a category $\mathcal{A}$. The* **right orthogonal to** *$\mathcal{B}$ is the full subcategory*

$$\mathcal{B}^{\perp} = \{A \in \mathcal{A} \mid \mathrm{Hom}_{\mathcal{A}}(B, A) = 0 \ \forall B \in \mathcal{B}\}.$$

Dually, the **left orthogonal to** *$\mathcal{B}$ is the full subcategory*

$${}^{\perp}\mathcal{B} = \{A \in \mathcal{A} \mid \mathrm{Hom}_{\mathcal{A}}(A, B) = 0 \ \forall B \in \mathcal{B}\}.$$

Definition 2.1.4.4. *Let $\mathcal{T}$ be a triangulated category. A full subcategory $\mathcal{S} \subset \mathcal{T}$ is* **extension-closed** *if whenever $X \to Y \to Z \to X[1]$ is a triangle in $\mathcal{T}$ with $X, Z \in \mathcal{S}$*

then $Y \in \mathcal{S}$. The **extension-closed subcategory of $\mathcal{T}$ generated by** $\mathcal{S}$ *is the smallest extension-closed full subcategory of $\mathcal{T}$ containing $\mathcal{S}$. We denote it by $\langle \mathcal{S} \rangle$.*

Definition 2.1.4.5. *A* **t-structure** *on a triangulated category $\mathcal{T}$ is a pair $(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})$ of strictly full subcategories of $\mathcal{T}$ such that by setting $\mathcal{T}^{\leq n} = \Sigma^{-n}\mathcal{T}^{\leq 0}$ and $\mathcal{T}^{\geq n} = \Sigma^{-n}\mathcal{T}^{\geq 0}$ for any $n \in \mathbb{Z}$ the following properties are satisfied:*

i) $\mathrm{Hom}_{\mathcal{T}}(X, Y) = 0$ *if $X \in \mathcal{T}^{\leq 0}$ and $Y \in \mathcal{T}^{\geq 1}$.*

ii) $\mathcal{T}^{\leq 0} \subset \mathcal{T}^{\leq 1}$.

iii) For any object $X \in \mathcal{T}$ there exists a distinguished triangle

$$A \longrightarrow X \longrightarrow B \longrightarrow \Sigma A$$

with $A \in \mathcal{T}^{\leq 0}$ and $B \in \mathcal{T}^{\geq 1}$.

Definition 2.1.4.6. *A t-structure $(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})$ on a triangulated category $\mathcal{T}$ is* **bounded** *if*

$$\mathcal{T} = \bigcup_{n \geq 0} \left(\mathcal{T}^{\leq n} \cap \mathcal{T}^{\geq n} \right).$$

Definition 2.1.4.7. *Let $\mathcal{T}$ and $\mathcal{D}$ be two triangulated categories equipped with t-structures $(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})$ and $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$. A functor $F : \mathcal{T} \to \mathcal{D}$ is* **left (resp right) t-exact** *if $F(\mathcal{T}^{\geq 0}) \subset \mathcal{D}^{\geq 0}$ (resp. if $F(\mathcal{T}^{\leq 0}) \subset \mathcal{D}^{\leq 0}$). We say that F is* **t-exact** *if it is both left and right t-exact.*

Proposition 2.1.4.8. *[BBD82, 1.3.3 and 1.3.5] Let $(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})$ be a t-structure on a triangulated category $\mathcal{T}$.*

i) The inclusion of the subcategory $\mathcal{T}^{\leq n}$ in $\mathcal{T}$ (respectively of $\mathcal{T}^{\geq n}$ in $\mathcal{T}$) has a right adjoint $\tau_{\leq n}$ (respectively a left adjoint $\tau_{\geq n}$).

ii) For any object $X \in \mathcal{T}$ there exists a unique $d \in \mathrm{Ext}^1_{\mathcal{T}}(\tau_{\geq 1}X, \tau_{\leq 0}X)$ such that the triangle

$$\tau_{\leq 0}X \longrightarrow X \longrightarrow \tau_{\geq 1}X \xrightarrow{d} \Sigma\tau_{\leq 0}X$$

is distinguished.

iii) For any $a \leq b$ and any $X \in \mathcal{T}$ there exists a unique isomorphism

$$\gamma : \tau_{\geq a}\tau_{\leq b}X \to \tau_{\leq b}\tau_{\geq a}X$$

such that the diagram

$$\begin{tikzcd} \tau_{\leq b}X \arrow[r] \arrow[d] & X \arrow[r] & \tau_{\geq a}X \\ \tau_{\geq a}\tau_{\leq b}X \arrow[rr, "\gamma"] & & \tau_{\leq b}\tau_{\geq a}X \arrow[u] \end{tikzcd}$$

is commutative.

We now present the notion of the heart of a t-structure which will turn out to be an abelian subcategory of the ambient triangulated category.

Definition 2.1.4.9. *Let $(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})$ be a t-structure on a triangulated category $\mathcal{T}$, then*

$$\mathcal{H} = \mathcal{T}^{\leq 0} \cap \mathcal{T}^{\geq 0}$$

is the **heart** *of the t-structure.*

Theorem 2.1.4.10. *[BBD82, 1.3.6] The heart $\mathcal{H}$ of a bounded t-structure $(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})$ on a triangulated category $\mathcal{T}$ is an abelian category stable under extensions.*

Proposition 2.1.4.11. *[BBD82, 1.3.6 and 1.3.7] Let $\mathcal{H}$ be the heart of a t-structure $(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})$ on a triangulated category $\mathcal{T}$.*

i) The functor $H^0 := \tau_{\geq 0}\tau_{\leq 0} : \mathcal{T} \to \mathcal{H}$ is cohomological.

ii) We have that $X \in \mathcal{T}^{\leq 0}$ (respectively $X \in \mathcal{T}^{\geq 0}$) if and only if $H^i(X) = 0$ for $i > 0$ (respectively $H^i(X) = 0$ for $i < 0$), where $H^i(X) := H^0(X[i])$.

Remark 2.1.4.12. *A bounded t-structure $(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})$ on a triangulated category $\mathcal{T}$ is completely determined by its heart $\mathcal{H}$, since*

$$\mathcal{T}^{\leq 0} = \langle \mathcal{H}, \Sigma\mathcal{H}, \Sigma^2\mathcal{H}, \ldots \rangle$$

is the extension-closure of the positive shifts of the heart and similarly

$$\mathcal{T}^{\geq 1} = \langle \Sigma^{-1}\mathcal{H}, \Sigma^2\mathcal{H}, \ldots \rangle.$$

Example 2.1.4.13. *([Dim04, Example 5.1.3]) Let* $\mathbf{D}(\mathcal{A})$ *be the bounded derived category of an abelian category* $\mathcal{A}$*. There is natural t-structure on* $\mathbf{D}(\mathcal{A})$ *given by*

$$\begin{aligned}\mathrm{Ob}(\mathcal{T}^{\leq 0}) &= \{\mathcal{K} \in \mathbf{D}(\mathcal{A}) : H^i(\mathcal{K}) = 0 \ \forall i > n\} \\ \mathrm{Ob}(\mathcal{T}^{\geq 0}) &= \{\mathcal{K} \in \mathbf{D}(\mathcal{A}) : H^i(\mathcal{K}) = 0 \ \forall i < n\},\end{aligned}$$

where $H : \mathbf{D}(\mathcal{A}) \to \mathcal{A}$ *denotes the standard cohomology functor.*

2.1.5 Torsion Pairs and HRS-tilting

In this section, we first recall the notion of faithful heart and discuss some related facts. Then, we introduce torsion pairs in an abelian category and generally in a triangulated category. The notion of torsion pair is closely related to the one of t-structure. Moreover, one can study new abelian categories inside the derived category as the product of a tilt at a torsion pair. The main references for this part are [HRS96] and [CHZ18].

Definition 2.1.5.1. *The heart* $\mathcal{H}$ *of a t-structure* $(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})$ *on a triangulated category* $\mathcal{T}$ *is* **faithful** *if there is an equivalence*

$$\mathbf{D}(\mathcal{H}) \to \mathcal{T}$$

restricting to an equivalence between the natural heart in $\mathbf{D}(\mathcal{H})$ *and the heart* $\mathcal{H}$ *in* $\mathcal{T}$*.*

Note that in general, there is not even a functor $\mathbf{D}(\mathcal{H}) \to \mathcal{T}$. The concept of faithful heart is very important as it gives a way to extract information about a triangulated category using data from an abelian subcategory.

Theorem 2.1.5.2. *[GM94, Chapter 5, Theorem 3.7.3] Let* $\mathcal{H}$ *be the heart of a bounded t-structure on a triangulated category* $\mathcal{T}$ *and* $F : \mathbf{D}^b(\mathcal{H}) \to \mathcal{T}$ *a t-exact functor. Then,* F *is an equivalence of categories if and only if* $\mathrm{Ext}^*_{\mathcal{T}}$ *is generated by* $\mathrm{Ext}^1_{\mathcal{H}}$*.*

Remark 2.1.5.3. *Note that, if the heart* $\mathcal{H}$ *is faithful we have that* $\mathrm{Ext}^k_{\mathcal{T}}(X, Y)$ *are generated by* $\mathrm{Ext}^1_{\mathcal{H}}(X, Y)$ *for any* $k \geq 0$*, therefore any* Ext*-group in* $\mathcal{T}$ *can be calculated in terms of Yoneda* Ext*-groups in the abelian category* $\mathcal{H}$*. On the other hand, if* $\mathcal{H}$ *is not faithful, we have natural isomorphisms*

$$\mathrm{Ext}^i_{\mathcal{T}}(X, Y) \cong \mathrm{Ext}^i_{\mathcal{H}}(X, Y) \ \forall i \leq 1$$

but not necessarily any obvious relation between these groups for $i \geq 2$.

We now recall a possible definition of a torsion pair in a triangulated category.

Definition 2.1.5.4. *A* **torsion pair** *in a triangulated category $\mathcal{T}$ is a pair of strictly full subcategories $(\mathcal{X}, \mathcal{Y})$ of $\mathcal{T}$ satisfying the following conditions:*

i) $\mathrm{Hom}_{\mathcal{T}}(\mathcal{X}, \mathcal{Y}) = 0$*, that is* $\mathrm{Hom}_{\mathcal{T}}(X, Y) = 0$ *for any $X \in \mathcal{X}$ and $Y \in \mathcal{Y}$.*

ii) $\Sigma\mathcal{X} \subseteq \mathcal{X}$ *and* $\Sigma^{-1}\mathcal{Y} \subseteq \mathcal{Y}$.

iii) For any object $T \in \mathcal{T}$ there exists a triangle (in $\mathcal{T}$)

$$X \to T \to Y \to \Sigma X$$

such that $X \in \mathcal{X}$ and $Y \in \mathcal{Y}$.
The subcategories $\mathcal{X}$ and $\mathcal{Y}$ are called the **torsion class** *and* **torsion-free class** *respectively.*

Note that in the literature one can find Definition 2.1.5.4 without condition ii) as the definition of a torsion pair in a triangulated category. The following proposition shows there is a very close relation between a torsion pair and a t-structure in a triangulated category $\mathcal{T}$.

Proposition 2.1.5.5. *[BR07, Proposition 2.13] Let $\mathcal{T}$ be a triangulated category. The maps*

$$\Phi : (\mathcal{X}, \mathcal{Y}) \to (\mathcal{X}, \Sigma\mathcal{Y})$$
$$\Psi : (\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0}) \to (\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 1})$$

are naturally inverse bijections between torsion pairs and t-structures in $\mathcal{T}$.

Similarly, one can also consider torsion pairs in an abelian category.

Definition 2.1.5.6. *A* **torsion pair** *in an abelian category $\mathcal{A}$ consists of a pair of full subcategories $(\mathscr{T}, \mathscr{F})$ such that:*

i) $\mathrm{Hom}_{\mathcal{A}}(\mathscr{T}, \mathscr{F}) = 0$*, that is* $\mathrm{Hom}_{\mathcal{A}}(T, F) = 0$ *for any $T \in \mathscr{T}$ and $F \in \mathscr{F}$.*

ii) For any object $A \in \mathcal{A}$ there exists a unique short exact sequence (up to isomorphism)

$$0 \to T \to A \to F \to 0$$

where $T \in \mathscr{T}$ and $F \in \mathscr{F}$.
The subcategories $\mathscr{T}$ and $\mathscr{F}$ are called the **torsion class** *and* **torsion-free class** *respectively.*

Remark 2.1.5.7. *Let $(\mathscr{T}, \mathscr{F})$ be a torsion pair in an abelian category $\mathcal{A}$, then, see discussion following [BR07, Definition 1.1]:*

i) $\mathscr{T}$ is closed under factors, extensions and coproducts while $\mathscr{F}$ is closed under sub-objects, extensions and products.

ii) We have $\mathscr{T}^{\perp} = \mathscr{F}$ and ${}^{\perp}\mathscr{F} = \mathscr{T}$.

iii) If $\mathcal{A}$ is locally small complete and cocomplete then any full subcategory of $\mathcal{A}$ closed under factors, extensions and coproducts is a torsion class while any full subcategory of $\mathcal{A}$ closed under sub-objects, extensions and products is a torsion-free class. The same holds under the milder assumption that $\mathcal{A}$ is noetherian, [LS18, Proposition 3.5].

We now recall the procedure of tilting, which allows one to produce a new heart starting from a torsion pair in a triangulated category.

Definition 2.1.5.8. *Let $\mathcal{A}$ be the heart of a t-structure on $\mathcal{T}$ and $(\mathscr{T}, \mathscr{F})$ a torsion pair in $\mathcal{A}$. Then the category*

$$\mathcal{B} = \{X \in \mathcal{T} \mid H^{-1}(X) \in \mathscr{F}, H^0(X) \in \mathscr{T}, H^i(X) = 0 \ \forall i \neq -1, 0\}$$

is the **forward HRS-tilt** *of $\mathcal{A}$ with respect to the torsion pair $(\mathscr{T}, \mathscr{F})$, where H^i are the cohomological functors associated to the t-structure with heart $\mathcal{A}$.*

Theorem 2.1.5.9. *[BR07, Theorem 3.1] There is a bijection between torsion pairs in $\mathcal{A}$ and hearts $\mathcal{B}$ with $\mathcal{B} \subset \langle \mathcal{A}, \Sigma\mathcal{A} \rangle$.*

In general, a heart of a t-structure need not to be faithful in the sense of Definition 2.1.5.1. Indeed, as pointed out in Remark 2.1.5.3, there might not be a way to compare the derived category of a heart to the ambient triangulated category. On the other hand, the following results gives equivalent conditions to have a derived equivalence between a faithful heart and its tilt at a torsion pair.

Remark 2.1.5.10. *Let $\mathcal{T}$ be a triangulated category with a bounded t-structure $(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})$ and heart $\mathcal{H}$. A* **realisation functor***, see [BBD82, Section 3.1], is a triangulated functor $G : \mathbf{D}(\mathcal{H}) \to \mathcal{T}$ which is t-exact and such that $G|_{\mathcal{H}} = \mathrm{id}_{\mathcal{H}}$. Such functor G exists when $\mathcal{T}$ is algebraic in the sense of [Kel06], that is when $\mathcal{T}$ is equivalent to the stable category of a Frobenius category. Examples of algebraic triangulated categories include homotopy and derived categories of abelian categories, see [Sch08]. Therefore, in our setting, the realisation functor G always exists.*

Theorem 2.1.5.11. *[CHZ18, Theorem A] Let $\mathcal{A}$ be an abelian category with a torsion pair $(\mathscr{T}, \mathscr{F})$. Let $\mathcal{B}$ be the forward HRS-tilt in $\mathbf{D}(\mathcal{A})$ and let $G : \mathbf{D}(\mathcal{A}) \to \mathbf{D}(\mathcal{B})$ be the realisation functor. Then the following conditions are equivalent:*

1) G is an equivalence.

2) $\mathcal{A}$ lies in the essential image of G.

3) Any object $A \in \mathcal{A}$ fits into an exact sequence of the form

$$0 \to F^0 \to F^1 \to A \to T^0 \to T^1 \to 0$$

with $F^i \in \mathscr{F}$ and $T^i \in \mathscr{T}$, such that the corresponding class in the third Yoneda extension group $\mathrm{Ext}^3_{\mathcal{A}}(T^1, F^0)$ vanishes.

2.2 Algebra and Representation Theory

In this section we recall some basic and well-known facts about algebras, quivers and briefly introduce Auslander-Reiten theory. The main references for this part are [ARS97, ASS06] as well as [DK12, Bar15, Sch14].

2.2.1 Finite Dimensional Algebras

We introduce the class of finite dimensional algebras together with some important properties, see [DK12, Bar15].

Definition 2.2.1.1. *Let $\Bbbk$ be a field, a* **$\Bbbk$-algebra** *is a ring* A *with an identity element, denoted by* 1*, such that* A *has a $\Bbbk$-vector space structure compatible with the multiplication. An algebra* A *is* **finite dimensional** *if the dimension of the $\Bbbk$-vector space* A *is finite.*

In what follows, for a $\Bbbk$-algebra A, we will denote by A-**Mod** the category of (right) A-modules and by A-**mod** the full subcategory of finitely generated (right) A-modules.

Remark 2.2.1.2. *Let* A *be an algebra. The category* A-**mod** *is a $\Bbbk$-linear category in the sense of Definition 2.1.1.5.*

Definition 2.2.1.3. *An element of an algebra $e \in \mathrm{A}$ is called* **idempotent** *if $e^2 = e$. Two idempotents $e, e' \in \mathrm{A}$ are* **orthogonal** *if $ee' = e'e = 0$. An idempotent $e \neq 0$ is* **primitive** *if for any two orthogonal idempotents $e', e'' \in \mathrm{A}$ with $e' + e'' = e$ we have either $e' = 0$ or $e'' = 0$. A set $\{e_1, \dots, e_n\}$ of pairwise orthogonal idempotents is called* **complete** *if $\sum_{i=1}^{n} e_i = 1$.*

Definition 2.2.1.4. *An algebra* A *is* **basic** *if there exists a complete set of pairwise orthogonal, primitive idempotents.*

Finite length $\Bbbk$-linear categories, see Definitions 2.1.1.5 and 2.1.1.13, are equivalent to a module category whenever they have a projective generator, see Definition 2.1.2.5. In particular, we have the following characterisation.

Proposition 2.2.1.5. *[Bas68, Chapter II, Exercise after Theorem 1.3] Let $\mathcal{C}$ be a finite length $\Bbbk$-linear abelian category (with finite dimensional morphism spaces). Then $\mathcal{C}$ has a projective generator if and only if there is an exact equivalence $\mathcal{C} \simeq A$-**mod** between $\mathcal{C}$ and the category of finite dimensional (right) modules over a finite dimensional $\Bbbk$-algebra* A.

Often it is helpful to study finite dimensional algebras such that the categories of modules over them are equivalent.

Definition 2.2.1.6. *Two finite dimensional algebras* A *and* B *are* **Morita equivalent** *if* A-**mod** $\simeq$ B-**mod**.

Remark 2.2.1.7. *Morita equivalence preserves finite dimensionality, in the sense that if* A *is finite dimensional and* B *is Morita equivalent to A, then* B *is also finite dimensional, see [AF92, Corollary 22.7].*

The next two propositions give important characterisations of finite dimensional algebras.

Proposition 2.2.1.8. *[Bar15, Proposition 3.16] For any finite dimensional algebra* A *there exists a basic finite dimensional algebra Morita equivalent to* A.

Proposition 2.2.1.9. *[ASS06, I, Proposition 6.2] Let* $\Bbbk$ *be an algebraically closed field and* A *a finite dimensional* $\Bbbk$*-algebra.*

i) A *is basic if and only if the algebra* $\mathrm{A}/\mathrm{rad}(A) \cong \Bbbk \times \Bbbk \times \ldots \times \Bbbk$.

ii) Every simple module over a basic $\Bbbk$*-algebra is one dimensional.*

We now introduce the notion of representation type of an algebra, which is given in terms of the number of (isomorphism classes of) indecomposable A-modules.

Definition 2.2.1.10. *A* $\Bbbk$*-algebra* A *has* **finite representation type** *if there are only finitely many finite dimensional indecomposable* A*-modules up to isomorphism. Otherwise,* A *is said to be of* **infinite representation type**.

Example 2.2.1.11. *Let* $\Bbbk$ *be an algebraically closed field.*

i) The algebra $\mathrm{A} = \Bbbk[x]/(x^n)$ *has finite representation type. Indeed, any* A*-module* M *is a vector space together with a linear map* $\phi : M \to M$ *such that* $\phi^n = 0$*. If we write* ϕ *as a matrix, the corresponding Jordan canonical form for* ϕ *is a block diagonal matrix where each block is a* $t \times t$ *matrix of the form*

$$J_t(0) = \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 1 \\ 0 & 0 & 0 & \ldots & 0 \end{pmatrix}$$

for some $t \leq n$*. Therefore,* M *is indecomposable if there is only one such block. Hence, there are exactly* n *isomorphism classes of indecomposable, one for each dimension.*

ii) The algebra $\mathrm{B} = \Bbbk[x, y]/(x^2, y^2)$ *has infinite representation type. Let* $M = \Bbbk^{2n}$ *for some* $n \geq 1$ *and pick* $\lambda \in \Bbbk$*. Let make* x *and* y *act by*

$$X = \begin{pmatrix} 0 & I_n \\ 0 & 0 \end{pmatrix} \qquad \text{and} \qquad Y = \begin{pmatrix} 0 & J_n(\lambda) \\ 0 & 0 \end{pmatrix}$$

where I_n is the $n \times n$ identity matrix and $J_n(\lambda) = J_n(0) + \lambda I_n$. Then, since $X^2 = Y^2 = 0$ and $XY = YX = 0$, this defines a B*-module. One can check that it is indecomposable and non-isomorphic for different values of n and λ.*

2.2.2 Quivers

In this section, we review some definitions about quivers and their representations, see [Sch14].

Definition 2.2.2.1. *A* **(finite) quiver** $\mathrm{Q} = (\mathrm{Q}_0, \mathrm{Q}_1)$ *is a directed graph where* Q_0 *is the (finite) set of vertices and* Q_1 *is the (finite) set of arrows. For an arrow $\alpha \in \mathrm{Q}_1$ of the form $\alpha : i \to j$ the source map $s : \mathrm{Q}_1 \to \mathrm{Q}_0$ and target map $t : \mathrm{Q}_1 \to \mathrm{Q}_0$ are defined as $s(\alpha) = i$ and $t(\alpha) = j$ respectively.*

Definition 2.2.2.2. *A quiver* Q *is* **connected** *if so is its underlying graph* $\overline{\mathrm{Q}}$.

Definition 2.2.2.3. *Let* Q *be a quiver, a* $\mathbb{k}$*-linear* **representation** *of* Q *is a pair $M = (M_{e_i}, \phi_\alpha)$ such that:*

i) For any $e_i \in \mathrm{Q}_0$ there is an associated $\mathbb{k}$-vector space M_{e_i}.

ii) for any arrow $\alpha \in \mathrm{Q}_1$ of the form $\alpha : e_i \to e_j$ there is an associated $\mathbb{k}$-linear map $\phi_\alpha : M_{e_i} \to M_{e_j}$.

A representation M is **finite dimensional** *if each M_{e_i} is a finite dimensional $\mathbb{k}$-vector space.*

Definition 2.2.2.4. *Let $M = (M_{e_i}, \phi_\alpha)$ and $M' = (M'_{e_i}, \phi'_\alpha)$ be two representations of a quiver* Q*. A* **morphism of representations** *$f : M \to M'$ is a family of $\mathbb{k}$-linear maps $f_{e_i} : M_{e_i} \to M'_{e_i}$ for $e_i \in \mathrm{Q}_0$ compatible with the maps ϕ_α, that is such that the diagram*

$$\begin{array}{ccc} M_{e_i} & \xrightarrow{\phi_\alpha} & M_{e_j} \\ {\scriptstyle f_{e_i}}\downarrow & & \downarrow{\scriptstyle f_{e_j}} \\ M'_{e_i} & \xrightarrow{\phi'_\alpha} & M'_{e_j} \end{array}$$

is commutative.

We denote by $\mathbf{Rep}(Q)$ the category of $\mathbb{k}$-linear representation of a quiver Q and by $\mathbf{rep}(Q) \subset \mathbf{Rep}(Q)$ the full subcategory consisting of finite dimensional representations of Q. In particular, we have the following result.

Lemma 2.2.2.5. *[ASS06, III.1.3] Let* Q *be a finite quiver. The categories* $\mathbf{Rep}(Q)$ *and* $\mathbf{rep}(Q)$ *are abelian.*

Definition 2.2.2.6. *Let* $\mathcal{A}$ *be a* $\Bbbk$*-linear category (with finitely many simple objects) such that* $\mathrm{Ext}^n_{\mathcal{A}}(A,B)$ *are finite dimensional for any* $A, B \in \mathcal{A}$ *and any* $n \in \mathbb{N}$*. The* Ext**-quiver of** $\mathcal{A}$ *has a vertex for each simple object* $S \in \mathcal{A}$ *and* $\dim \mathrm{Ext}^1_{\mathcal{A}}(S_i, S_j)^\vee$ *arrows from the vertex labelled by* S_i *to the vertex labelled by* S_j*.*

The following Lemma gives a criterion to check if a representation of a quiver is indecomposable.

Lemma 2.2.2.7. *[EH18, Lemma 9.11] Let* $M \in \mathbf{rep}(Q)$*.* M *is indecomposable if and only if the only homomorphisms of representations* $\phi : M \to M$ *such that* $\phi^2 = \phi$ *are the zero homomorphism and the identity.*

Given a quiver Q, we can construct an algebra from it.

Definition 2.2.2.8. *A* **path** *in a quiver* Q *is either a sequence of arrows* $\alpha_1 \dots \alpha_n$ *such that* $t(\alpha_i) = s(\alpha_{i+1})$ *for each* $1 \leq i \leq n$ *or a vertex* $e_i \in \mathrm{Q}_0$ *which is called a* **trivial path***.*

Definition 2.2.2.9. *The* **path algebra of a quiver** Q *over a field* $\Bbbk$ *is the* $\Bbbk$*-vector space of all paths in* Q *with multiplication given by concatenation of paths, that is if* p *and* q *are paths in* Q *then*

$$p \cdot q = \begin{cases} pq & \text{if } s(q) = t(p) \\ 0 & \text{otherwise} \end{cases}$$

where one extends the functions s *and* t *to the set of all paths.*

The next result gives a characterisation of finite representation type path algebras.

Theorem 2.2.2.10 (Gabriel)**.** *Let* Q *be a finite quiver. Then,* $\Bbbk\mathrm{Q}$ *has finite representation type if and only if* $\overline{\mathrm{Q}}$ *is a disjoint union of Dynkin diagrams of type ADE.*

Definition 2.2.2.11. *Let* Q *be a finite connected quiver, the (two-sided) ideal* $\mathrm{R_Q}$ *of* $\Bbbk\mathrm{Q}$ *generated by* Q_1 *is called the* **arrow ideal***.*

Definition 2.2.2.12. *Let* Q *be a finite quiver and* $\mathrm{R_Q}$ *the arrow ideal of* $\Bbbk\mathrm{Q}$*. A two-sided ideal* I *of* $\Bbbk\mathrm{Q}$ *is* **admissible** *if there exists* $m \geq 2$ *such that*

$$\mathrm{R}^m_{\mathrm{Q}} \subseteq \mathrm{I} \subseteq \mathrm{R}^2_{\mathrm{Q}}.$$

Definition 2.2.2.13. *Let* Q *be a quiver, a* **relation** ρ *in* Q *with coefficients in* $\mathbb{k}$ *is a* $\mathbb{k}$*-linear combination of paths, that is an element in* $\mathbb{k}$Q *of the form*

$$\rho = \sum_{i=1}^{m} \lambda_i p_i$$

where $p_i \in \mathrm{R}^2_{\mathrm{Q}}$ *and* $\lambda_i \in \mathbb{k}$ *are not all zero.*
We denote a quiver with its ideal of relations by (Q, I) *and by* $\mathbf{rep}(\mathrm{Q}, \mathrm{I})$ *the category of its finite dimensional representations.*

Let Q be a quiver with I an admissible ideal of $\mathbb{k}$Q. One can quotient the path algebra of the quiver Q by such ideal. The next result characterises the quotient algebra $\mathbb{k}\mathrm{Q}/\mathrm{I}$ and gives an equivalence between the category of finitely generated modules over such an algebra and the category of representations of the quiver with relations Q.

Theorem 2.2.2.14. *[ASS06, II.2.12 and III.1.6] Let* Q *be a quiver and* I *an admissible ideal of the path algebra* $\mathbb{k}$Q*. Then* $\mathrm{A} = \mathbb{k}\mathrm{Q}/\mathrm{I}$ *is a basic connected finite dimensional algebra such that there exists an equivalence of categories*

$$A\text{-}\mathbf{mod} \xrightarrow{\simeq} \mathbf{rep}(\mathrm{Q}, \mathrm{I}).$$

2.2.3 Auslander-Reiten Theory

In this section we give a short introduction to Auslander-Reiten theory, introduced by Maurice Auslander and Idun Reiten in 1975. This theory studies the representation theory of a class of algebras which contains finite dimensional algebras. The main references for this part are [ARS97, ASS06]. We start with the definition of an Artin algebra, a class of algebras for which Auslander-Reiten theory can be carried out in full generality.

Definition 2.2.3.1. *Let* R *be a ring. An* R*-algebra* Λ *is an* **Artin algebra** *if it is finitely generated as* R*-module.*

Note that finite dimensional algebras over a field are examples of Artin algebras. In the rest of this section, we assume that Λ is an Artin algebra even though we will be mainly interested in the case of finite dimensional algebras.

Definition 2.2.3.2. *A morphism* $f \in \mathrm{Hom}_{\Lambda\text{-}\mathbf{mod}}(L, M)$ *is called:*

i) **Left minimal** *if every* $h \in \mathrm{End}(M)$ *such that* $hf = f$ *is an automorphism.*

ii) **Left almost split** *if f is not a section and for every $u \in \mathrm{Hom}_{\mathbf{\Lambda}\text{-}\mathbf{mod}}(L, N)$ there exists $u' \in \mathrm{Hom}_{\mathbf{\Lambda}\text{-}\mathbf{mod}}(M, N)$ such that $u'f = u$, that is such that the diagram*

$$
\begin{tikzcd}
L \arrow[r, "f"] \arrow[d, "u"'] & M \arrow[dl, "u'"] \\
U &
\end{tikzcd}
$$

commutes.

iii) **Left minimal almost split** *if it is both left minimal and almost split.*

Dually, one can define **right minimal**, **right almost split** *and* **right minimal almost split** *respectively, see [ASS06, IV Definition 1.1].*

Almost split morphisms are very closely related to indecomposable objects in **Λ-mod**.

Lemma 2.2.3.3. *[ASS06, IV Lemma 1.3] Let $f \in \mathrm{Hom}_{\mathbf{\Lambda}\text{-}\mathbf{mod}}(M, N)$.*

i) If f is left almost split then $M \in$ **Λ-mod** *is indecomposable.*

ii) If f is right almost split then $N \in$ **Λ-mod** *is indecomposable.*

We can now introduce the notion of irreducible morphism in **Λ-mod**.

Definition 2.2.3.4. *A morphism $f \in \mathrm{Hom}_{\mathbf{\Lambda}\text{-}\mathbf{mod}}(M, N)$ is* **irreducible** *if*

i) f is neither a section nor a retraction and

ii) if $f = f_1 \circ f_2$ for $f_1 \in \mathrm{Hom}_{\mathbf{\Lambda}\text{-}\mathbf{mod}}(M, L)$ and $f_2 \in \mathrm{Hom}_{\mathbf{\Lambda}\text{-}\mathbf{mod}}(L, N)$, then either f_1 is a retraction or f_2 is a section.

Remark 2.2.3.5. *Let $f \in \mathrm{Hom}_{\mathbf{\Lambda}\text{-}\mathbf{mod}}(M, N)$ be an irreducible morphism. Then f is either a proper monomorphism or a proper epimorphism.*

From the projective cover and injective hull of a simple object in **Λ-mod**, one can construct examples of irreducible morphisms.

Lemma 2.2.3.6. *[ASS06, IV Example 1.5] Let P the projective cover of a simple object S in* **Λ-mod***. Then the morphism*

$$\mathrm{rad}(P) \hookrightarrow P$$

is both right almost split and irreducible. Dually, if I is the injective hull of S in **Λ-mod**, *the morphism*

$$I \twoheadrightarrow I/S$$

is left almost split and irreducible.

We now give a characterisation of irreducible morphisms in **Λ-mod** and of kernels and cokernels of an irreducible morphism.

Lemma 2.2.3.7. *[ASS06, Chapter IV Lemma 1.7] Let $0 \to L \xrightarrow{f} M \xrightarrow{g} N \to 0$ be a non-split short exact sequence in* **Λ-mod**.

i) The morphism $f : L \to M$ is irreducible if and only if for every morphism $v : V \to N$ there exists either $v_1 : V \to M$ such that $v = g \circ v_1$ or there exists $v_2 : M \to V$ such that $g = v \circ v_2$.

ii) The morphism $g : M \to N$ is irreducible if and only if for every morphism $u : L \to U$ there exists either $u_1 : M \to U$ such that $u = u_1 \circ f$ or there exists $u_2 : U \to M$ such that $f = u_2 \circ u$.

It turns out that kernels and cokernels of irreducible morphisms give rise to other indecomposable objects.

Corollary 2.2.3.8. *[ASS06, Chapter IV Corollary 1.8] Let $f : M \to N$ be an irreducible morphism in* **Λ-mod**.

i) If f is a monomorphism, then $\mathrm{coker} f \in$ **Λ-mod** *is indecomposable.*

ii) If f is an epimorphism, then $\ker f \in$ **Λ-mod** *is indecomposable.*

We now present the notion of the radical between two indecomposable objects in **Λ-mod** and relate this notion to irreducible morphisms.

Definition 2.2.3.9. *Let $M, N \in$* **Λ-mod** *be indecomposable, then* $\mathrm{rad}_{\mathbf{\Lambda\text{-mod}}}(M, N)$ *is the $\Bbbk$-vector space of all non-invertible homomorphisms from M to N.*

Lemma 2.2.3.10. *[ASS06, IV Lemma 1.6] Let $M, N \in$* **Λ-mod** *be indecomposable, then $f \in \mathrm{Hom}_{\mathbf{\Lambda\text{-mod}}}(M, N)$ is irreducible if and only if $f \in \mathrm{rad}_{\mathbf{\Lambda\text{-mod}}}(M, N) \backslash \mathrm{rad}^2_{\mathbf{\Lambda\text{-mod}}}(M, N)$.*

Remark 2.2.3.11. *Lemma 2.2.3.10 implies that the space* $\mathrm{rad}_{\mathbf{\Lambda\text{-mod}}}(M, N)/\mathrm{rad}^2_{\mathbf{\Lambda\text{-mod}}}(M, N)$ *measures the number of indecomposable morphisms between two indecomposable objects $M, N \in$* **Λ-mod**, *therefore we use the notation* $\mathrm{Irr}(M, N) = \mathrm{rad}_{\mathbf{\Lambda\text{-mod}}}(M, N)/\mathrm{rad}^2_{\mathbf{\Lambda\text{-mod}}}(M, N)$.

We now introduce almost split sequences, which are crucial in Auslander-Reiten theory.

Definition 2.2.3.12. *A short exact sequence*

$$0 \to L \xrightarrow{f} M \xrightarrow{g} N \to 0$$

in **Λ-mod** *is called* **almost split sequence** *if the following conditions hold:*

i) f is left minimal almost split.

ii) g is right minimal almost split.

Almost split sequences are also called **Auslander-Reiten sequences** *or* **AR sequences** *for short.*

Remark 2.2.3.13. *[ASS06, IV Theorem 1.13] The first and last term in an almost split sequence are indecomposable. Moreover, the fact that it does not split means that the first term is not injective and last is not projective. Finally, an almost split sequence is determined by each of its end terms.*

We can consider the Λ-dual functor

$$(-)^t = \mathrm{Hom}_{\mathbf{\Lambda\text{-mod}}}(-, \Lambda) : \mathbf{\Lambda\text{-mod}} \to \mathbf{\Lambda\text{-mod}}^{op}.$$

We will use the fact that one can identify $\mathbf{\Lambda\text{-mod}}^{op}$ with $\mathbf{\Lambda^{op}\text{-mod}}$, see [ARS97, V.1.14 and V.1.15]. It is easy to note that if $P \in \mathbf{\Lambda\text{-mod}}$ is a projective right Λ-module then $P^t = \mathrm{Hom}_{\mathbf{\Lambda\text{-mod}}}(P, \Lambda)$ is a projective left Λ-module. Indeed, the above functor $(-)^t$ induces a duality between projective right Λ-modules and projective left Λ-modules. Let $M \in \mathbf{\Lambda\text{-mod}}$ and consider a minimal projective presentation

$$P_1 \xrightarrow{p_1} P_0 \xrightarrow{p_0} M \to 0.$$

Applying the left exact contravariant functor $(-)^t$ we obtain an exact sequence of the form

$$0 \to M^t \xrightarrow{p_0^t} P_0^t \xrightarrow{p_1^t} P_1^t \to \mathrm{coker}(p_1^t) \to 0.$$

in $\mathbf{\Lambda\text{-mod}}^{op}$. We set $\mathrm{Tr}M = \mathrm{coker}(p_1^t)$ and call it the **transpose of** M.

Proposition 2.2.3.14. *[ASS06, IV Proposition 2.1] Let $M \in$* **Λ-mod** *be indecomposable:*

1) $\mathrm{Tr}M \in \mathbf{\Lambda\text{-}mod}^{op}$ *has no non-zero projective direct summands.*

2) If M *is non-projective, the sequence*

$$P_0^t \xrightarrow{p_1^t} P_1^t \to \mathrm{Tr}M \to 0$$

induced from the minimal projective presentation of M *is a minimal projective presentation of the left* Λ*-module* $\mathrm{Tr}M$*.*

3) M *is projective if and only if* $\mathrm{Tr}M = 0$*.*

4) If M *is not projective then* $\mathrm{Tr}M$ *is indecomposable and* $\mathrm{Tr}(\mathrm{Tr}M) \cong M$*.*

5) If $M, N \in \mathbf{\Lambda\text{-}mod}$ *are indecomposable non-projective, then* $M \cong N$ *if and only if* $\mathrm{Tr}M \cong \mathrm{Tr}N$*.*

Remark 2.2.3.15. *The transpose* Tr *does not define a duality between* $\mathbf{\Lambda\text{-}mod}$ *and* $\mathbf{\Lambda\text{-}mod}^{op}$ *since it is zero on projective objects.*

In order to restore such duality, let $M, N \in \mathbf{\Lambda\text{-}mod}$ and consider the ideal of $\mathbf{\Lambda\text{-}mod}$ given by

$$\mathcal{P}(M,N) = \{f \in \mathrm{Hom}_{\mathbf{\Lambda\text{-}mod}}(M,N) \mid f \text{ factors through a projective } \Lambda\text{-module}\}.$$

Definition 2.2.3.16. *The quotient category*

$$\underline{\mathbf{\Lambda\text{-}mod}} = \mathbf{\Lambda\text{-}mod}/\mathcal{P}$$

is the **projectively stable category***. It has the same objects as* $\mathbf{\Lambda\text{-}mod}$ *while morphisms are defined as the quotient* $\mathbb{k}$*-vector spaces*

$$\underline{\mathrm{Hom}}_{\mathbf{\Lambda\text{-}mod}}(M,N) = \mathrm{Hom}_{\mathbf{\Lambda\text{-}mod}}(M,N)/\mathcal{P}(M,N) \quad \forall M, N \in \mathbf{\Lambda\text{-}mod}$$

with composition induced from the composition in $\mathbf{\Lambda\text{-}mod}$*. Dually one can consider the* $\mathbb{k}$*-subspace* $\mathcal{I}(M,N)$ *of* $\mathrm{Hom}_{\mathbf{\Lambda\text{-}mod}}(M,N)$ *of homomorphisms factoring through an injective* Λ*-module and consider the quotient category*

$$\overline{\mathbf{\Lambda\text{-}mod}} = \mathbf{\Lambda\text{-}mod}/\mathcal{I}$$

called the **injectively stable category**. *The objects of* $\overline{\mathbf{\Lambda}\text{-}\mathbf{mod}}$ *are the same as the objects of* $\mathbf{\Lambda}$**-mod**, *while morphisms in* $\overline{\mathbf{\Lambda}\text{-}\mathbf{mod}}$ *are defined as the quotient* $\mathbb{k}$*-vector spaces*

$$\overline{\mathrm{Hom}}_{\mathbf{\Lambda}\text{-}\mathbf{mod}}(M,N) = \mathrm{Hom}_{\mathbf{\Lambda}\text{-}\mathbf{mod}}(M,N)/\mathcal{I}(M,N).$$

Proposition 2.2.3.17. *[ASS06, IV Proposition 2.2] There is a* $\mathbb{k}$*-linear duality*

$$\begin{aligned} \underline{\mathbf{\Lambda}\text{-}\mathbf{mod}} &\longrightarrow \underline{\mathbf{\Lambda}\text{-}\mathbf{mod}}^{op} \\ M &\mapsto \mathrm{Tr}M \end{aligned}$$

called **transposition**.

One can consider another duality, namely

$$\mathcal{D} = \mathrm{Hom}_{\mathbb{k}}(-,\mathbb{k}) : \mathbf{\Lambda}\text{-}\mathbf{mod} \to \mathbf{\Lambda}\text{-}\mathbf{mod}^{op}.$$

Definition 2.2.3.18. *The* **Nakayama functor** *is defined as the composition.*

$$\nu = \mathcal{D}(-)^t = \mathcal{D}\mathrm{Hom}_{\mathbf{\Lambda}\text{-}\mathbf{mod}}(-,\Lambda) : \mathbf{\Lambda}\text{-}\mathbf{mod} \to \mathbf{\Lambda}\text{-}\mathbf{mod}$$

The Nakayama functor induces an equivalence of categories between projective Λ-modules and injective Λ-modules. In particular $\nu^{-1} = \mathrm{Hom}_{\mathbf{\Lambda}\text{-}\mathbf{mod}}(\mathcal{D}\Lambda,-)$ is quasi-inverse to ν.

Definition 2.2.3.19. *[ASS06, IV Proposition 2.4] Let* $P_1 \xrightarrow{p_1} P_0 \xrightarrow{p_0} M \to 0$ *be a minimal projective presentation of* $M \in \mathbf{\Lambda}$**-mod**. *There exists an exact sequence*

$$0 \to \tau M \to \nu P_1 \xrightarrow{\nu p_1} \nu P_0 \xrightarrow{\nu p_0} \nu M \to 0$$

in $\mathbf{\Lambda}$**-mod** *where* τM *is the* **Auslander-Reiten translation of** M. *Dually, given a minimal injective presentation* $0 \to N \xrightarrow{i_0} E_0 \xrightarrow{i_1} E_1$ *of* $N \in \mathbf{\Lambda}$**-mod** *there exists an exact sequence*

$$0 \to \nu^{-1}N \xrightarrow{\nu^{-1}i_0} \nu^{-1}E_0 \xrightarrow{\nu^{-1}i_1} \nu^{-1}E_1 \to \tau^{-1}N \to 0$$

in $\mathbf{\Lambda}$**-mod** *where* $\tau^{-1}N$ *is the* **inverse Auslander-Reiten translation of** N.

We have the following dual characterisations of the Auslander-Reiten translations.

Proposition 2.2.3.20. *[ASS06, IV Proposition 2.10] Let $M, N \in$ **Λ-mod** be indecomposable:*

*1) $\tau M \in$ **Λ-mod** is zero if and only if M is projective.*

2) If M is non-injective, then τM is indecomposable and $\tau^{-1}\tau M \cong M$.

3) If M and N are non-injective, then $M \cong N$ if and only if $\tau M \cong \tau N$.

Dually, we have:

*1') $\tau^{-1} N \in$ **Λ-mod** is zero if and only if N is injective.*

2') If N is non-injective, then $\tau^{-1}N$ is indecomposable and $\tau\tau^{-1}N \cong N$.

3') If M and N are non-injective, then $M \cong N$ if and only if $\tau^{-1}M \cong \tau^{-1}N$.

Proposition 2.2.3.21. *[ASS06, IV Corollary 2.11] The Auslander-Reiten translation induces and equivalence*

$$\tau : \underline{\mathbf{\Lambda\text{-mod}}} \to \overline{\mathbf{\Lambda\text{-mod}}}$$

with inverse τ^{-1}.

Theorem 2.2.3.22 (Auslander-Reiten Formulas)**.** *[ASS06, IV Theorem 2.13] Let $M, N \in$* **Λ-mod**. *There exist isomorphisms*

$$\mathcal{D}\underline{\mathrm{Hom}}_{\mathbf{\Lambda\text{-mod}}}(\tau^{-1}N, M) \cong \mathrm{Ext}^1_{\mathbf{\Lambda\text{-mod}}}(M, N) \cong \mathcal{D}\overline{\mathrm{Hom}}_{\mathbf{\Lambda\text{-mod}}}(N, \tau M)$$

functorial in both variables.

The existence of almost split sequences is given by the following:

Theorem 2.2.3.23. *[ASS06, IV Theorem 3.1] Let $M, N \in$ **Λ-mod** be indecomposable:*

*i) If M is non-projective, there exists an almost split sequence in **Λ-mod** of the form*

$$0 \to \tau M \to E \to M \to 0.$$

*ii) If M is non-injective, there exists an almost split sequence in **Λ-mod** of the form*

$$0 \to N \to F \to \tau^{-1}N \to 0.$$

Finally, when the Auslander-Reiten translations exist, the number of irreducible maps between two objects is the same as the number of maps between the corresponding translations.

Lemma 2.2.3.24. *[ASS06, IV Corollary 4.1] Let $M, N \in \mathbf{\Lambda}$-**mod** be indecomposable obejcts.*

i) If $\tau M \neq 0$ and $\tau N \neq 0$, then there exists a $\Bbbk$-linear isomorphism $\operatorname{Irr}(\tau M, \tau N) \cong \operatorname{Irr}(M, N)$.

ii) If $\tau^{-1} M \neq 0$ and $\tau^{-1} N \neq 0$, then there exists a $\Bbbk$-linear isomorphism $\operatorname{Irr}(\tau^{-1} M, \tau^{-1} N) \cong \operatorname{Irr}(M, N)$.

We can now define the Auslander-Reiten quiver a category of modules over an Artin algebra.

Definition 2.2.3.25. *The* **Auslander-Reiten quiver** $\Gamma(\mathbf{\Lambda}\text{-}\mathbf{mod})$ *of the category* $\mathbf{\Lambda}$-**mod** *is defined as follows:*

i) The vertices are the isomorphism classes of indecomposable Λ-modules.

ii) Given two vertices $[M]$ and $[N]$, the arrows $[M] \to [N]$ are in one-to-one correspondence with the vectors of a basis of the $\Bbbk$-vector space $\operatorname{rad}_{\mathbf{\Lambda}\text{-}\mathbf{mod}}(M, N)/\operatorname{rad}^2_{\mathbf{\Lambda}\text{-}\mathbf{mod}}(M, N)$.

iii) Given two vertices $[M]$ and $[N]$ there is a dashed arrow $[M] \dashrightarrow [N]$ if $\tau M \cong N$.

Remark 2.2.3.26. *Given a $\Bbbk$-linear category $\mathcal{C}$ equivalent to a module category, see Proposition 2.2.1.5, one can define the* **Ausander-Reiten quiver** *of $\mathcal{C}$ in terms of its (isomorphism classes of) indecomposable objects and indecomposable morphisms.*

2.2.4 On Irreducible Morphisms

In this section we give a concrete explanation on how to find irreducible morphisms, mainly based on the proof of Lemma 2.2.3.7. As pointed out in Remark 2.2.3.26, the interest in this class of morphisms is due to the fact that those morphisms represent the arrows in the Auslander-Reiten quiver of a category $\mathcal{C} \simeq A$-**mod**.

Recall that, if $\alpha : M \to N$ is irreducible in A-**mod**, then $M, N \in A$-**mod** are indecomposable. Moreover, every irreducible morphism is either a proper monomorphism or a proper epimorphism, see Remark 2.2.3.5. The easiest way to find an irreducible morphism is given by Lemma 2.2.3.6. If $\alpha : M \to N$ is neither a monomorphism nor an epimorphism, it is automatically reducible. In order to find irreducible morphisms involving M and N one can consider the factorisations

$$\begin{tikzcd} 0 \arrow[r] & \ker\alpha \arrow[r] & M \arrow[rr, "\alpha"] \arrow[dr, two heads] & & N \arrow[r] & \mathrm{coker}\alpha \arrow[r] & 0 \\ & & & \mathrm{im}\alpha \arrow[ur, hook] & & & \end{tikzcd},$$

that is one can consider the two short exact sequences

$$0 \to \ker\alpha \to M \to \mathrm{im}\alpha \to 0$$
$$0 \to \mathrm{im}\alpha \to N \to \mathrm{coker}\alpha \to 0.$$

Lemma 2.2.3.7 gives a way to check if a certain monomorphism or epimorphism is irreducible.

Remark 2.2.4.1. *Let $M, N \in A$-**mod** be indecomposable and suppose that $\alpha : M \twoheadrightarrow N$ is a non-split epimorphism. One can consider the exact sequence in A-**mod** induced by α, that is*

$$0 \to \ker\alpha \xrightarrow{\alpha'} M \xrightarrow{\alpha} N \to 0.$$

*One can use Lemma 2.2.3.7 to check if α is irreducible. If that is the case, by Corollary 2.2.3.8 we have that $\ker\alpha \in A$-**mod** is indecomposable and the monomorphism α' is non-split (because so is α). Therefore in this case, one can use again Lemma 2.2.3.7 to check if α' is irreducible. Note that the situation of $\alpha : M \hookrightarrow N$ non-split monomorphism is completely dual. That is, if α is irreducible one should use Lemma 2.2.3.7 to check if $\alpha'' : N \twoheadrightarrow \mathrm{coker}\alpha$ is irreducible.*

If it turns out that if α is reducible, that is if α admits a non-trivial factorisation, we can explain how to refine the search for irreducible morphisms. Starting from a non-split morphism, the following Remark gives a way to construct maps which one can then check are irreducible using Lemma 2.2.3.7.

Remark 2.2.4.2. *Let us assume that* A *is an algebra of finite representation type. Let us suppose that $M, N \in A$-**mod** are indecomposable objects and $\alpha : M \twoheadrightarrow N$ is a non-split* *reducible epimorphism. Thus, α admits a non-trivial factorisation*

$$\begin{tikzcd} M \arrow[r, two heads, "\alpha"] \arrow[d, "\sigma"'] & N \arrow[d, equal, "\mathrm{id}_N"] \\ L \arrow[r, two heads, "\mu"] & N \end{tikzcd}$$

*where $L \in A$-**mod** is indecomposable (as we are considering the decomposition of α as a chain of irreducible morphisms) and μ can be chosen to be an irreducible epimorphism (since it is the last map in the chain of irreducible morphisms from M to N) and σ non-split (as composite of irreducible maps). One can then complete the irreducible morphism μ to a non-split short exact sequence, where* $\ker \mu$ *is indecomposable by Corollary 2.2.3.8, and take the pullback along $\psi : \ker \mu \to L$ to obtain:*

$$\begin{tikzcd} 0 \arrow[r] & \ker \alpha \arrow[r, dashed, "\beta"] \arrow[d, dashed, "\phi"] & M \arrow[r, "\alpha"] \arrow[d, "\sigma"] & N \arrow[r] \arrow[d, equal, "\mathrm{id}_N"] & 0 \\ 0 \arrow[r] & \ker \mu \arrow[r, "\psi"] & L \arrow[r, "\mu"] & N \arrow[r] & 0 \end{tikzcd} . \tag{2.1}$$

Note that in (2.1) β and ψ are non-split monomorphisms as α and μ are non-split epimorphisms. We can now extend diagram (2.1) and consider

$$\begin{tikzcd} & \ker \phi \arrow[r, "\theta"] \arrow[d, hook] & \ker \sigma \arrow[d, hook, "\tau"] & & \\ 0 \arrow[r] & \ker \alpha \arrow[r, "\beta"] \arrow[d, dashed, "\phi"] & M \arrow[r, "\alpha"] \arrow[d, "\sigma"] & N \arrow[r] \arrow[d, equal, "\mathrm{id}_N"] & 0 \\ 0 \arrow[r] & \ker \mu \arrow[r, "\psi"] \arrow[d, two heads] & L \arrow[r, "\mu"] \arrow[d, two heads] & N \arrow[r] & 0 \\ & \mathrm{coker}\phi \arrow[r, dashed, "\xi"] & \mathrm{coker}\sigma & & \end{tikzcd} \tag{2.2}$$

where θ and ξ are isomorphisms by the Snake Lemma. There are three cases:

i) If σ is neither a monomorphism nor an epimorphism, then it cannot be irreducible.

ii) If σ is a monomorphism then ϕ is a non-split monomorphism. Moreover we have $\mathrm{coker}\phi \cong \mathrm{coker}\sigma$ *and* $\ker \phi \cong \ker \sigma \cong 0$ *and ψ cannot split as* $\ker \mu$ *is indecomposable. Furthermore, ϕ cannot split as if it did there would exist $\eta : \mathrm{coker}\phi \hookrightarrow \ker \mu$ such that one can write $L \twoheadrightarrow \mathrm{coker}\sigma$ as $\psi \circ \eta$, forcing ψ to be an isomorphism, which is* *impossible. Therefore, one should look at the non-split short exact sequences*

$$0 \to \ker\alpha \xrightarrow{\phi} \ker\mu \to \mathrm{coker}\phi \to 0$$
$$0 \to M \xrightarrow{\sigma} L \to \mathrm{coker}\phi \to 0$$

in order to refine the search for irreducible morphisms.

iii) If σ is an epimorphism, then ϕ is a non-split epimorphisms by the Snake Lemma we have $\mathrm{coker}\sigma \cong \mathrm{coker}\mu \cong 0$*. In addition, it is non-split as if ϕ splits, there exists* $\rho : \ker\alpha \twoheadrightarrow \ker\phi$*. The fact that we can write the monomorphism $\beta = \tau \circ \rho$ implies that ρ is a monomorphism, hence ρ is an isomorphism and* $\ker\mu \cong 0$*, which is impossible as μ is a non-split epimorphism between different indecomposable objects. Therefore, one should look at the non-split short exact sequences*

$$0 \to \ker\phi \to \ker\alpha \to \ker\mu \to 0$$
$$0 \to \ker\phi \to M \xrightarrow{\sigma} L \to 0$$

Finally, one should also check if in the non-split short exact sequence

$$0 \to \ker\mu \to L \xrightarrow{\mu} N \to 0$$

the first morphism (which is a non-split monomorphism) is irreducible.

Remark 2.2.4.3. *Note that if* A *is a finite representation type algebra, the process of finding irreducible morphisms described above must end, that is if one starts with an irreducible morphism and applies the above procedure one gets at least one irreducible morphism between objects which are shorter in length then the ones involved in the starting morphisms.*

2.3 Topology

In this section, we introduce the abelian category of p-perverse sheaves on a topologically stratified space. Perverse sheaves were introduced in the monograph 'Faisceaux Pervers' by Alexander Beilinson, Joseph Bernstein and Pierre Deligne in 1983. By choosing a perversity, that is a $\mathbb{Z}$-valued function having the set of strata as domain, one define a perverse t-structure on the constructible derived category. The heart of such perverse t-structure defines the abelian category of p-perverse sheaves. The fact that perverse sheaves are an abelian subcategory of the constructible derived category makes their study simpler, as the tools introduced in Section 2.1.1 can be used. Perverse sheaves turn out to be closely related to intersection cohomology, introduced in the same period by Mark Goresky and Robert MacPherson, and coincides with constructible sheaves for the case of the zero perversity. The main reference for this part is [BBD82] as well as [Dim04, GM80, KS13, HT07].

2.3.1 Topologically Stratified Spaces

In this section we introduce the class of spaces we are interested in. Indeed, we give the inductive definition of a topologically stratified space in the sense of [GM80, GM83] and we provide some examples.

Definition 2.3.1.1. *A* **zero dimensional topologically stratified space** *is a countable set of points with the discrete topology. An* n**-dimensional topologically stratified space** X *is a paracompact Hausdorff topological space endowed with a finite filtration by closed subsets of the form*

$$X = X_n \supset \ldots \supset X_0 \supset X_{-1} = \emptyset$$

such that:

1) *Every* $X_i \smallsetminus X_{i-1}$ *is a (possibly empty) topological* i*-manifold.*

2) *For every* $x \in X_i \smallsetminus X_{i-1}$ *there exist an open neighbourhood* $U \subset X$ *and a compact* $(n-i-1)$*-dimensional topologically stratified space* $L = L_{n-i-1}$*, called the* **link***, with stratification*

$$L = L_{n-i-1} \supset \ldots \supset L_0 \supset L_{-1} = \emptyset$$

and a filtration preserving homeomorphism

$$\phi : U \xrightarrow{\cong} \mathbb{R}^i \times C(L)$$

where $C(L) = L \times [0,1)/L \times \{0\}$ *is the open cone on* L *with the induced filtration by the vertex and the subsets* $L_i \times [0,1)/L_i \times \{0\}$*.*

The connected components of $X_i \smallsetminus X_{i-1}$ *are the* **strata** *of* X*.*

Remark 2.3.1.2. *Although condition 1) in Definition 2.3.1.1 is usually given as a consequence of the definition of topologically stratified space, see [GM83, 1.1], we included it in order to underline the fact that each successive difference in the filtration gives rise to a (possibly) empty topological manifold. Note that this is the case when $X_i = X_{i+1}$ in the filtration, see Example 2.3.1.6.*

Remark 2.3.1.3. *Let X be a topologically stratified space. The set of strata of X satisfy the frontier condition, that is*

$$S \cap \overline{T} \neq \emptyset \iff S \subset \overline{T}.$$

Moreover, there is a partial order on the set of strata of X given by

$$S \leq T \iff S \subset \overline{T}.$$

In fact, one can note that if $S \cap \overline{T} \neq \emptyset$, then from the local description of X as $\mathbb{R}^d \times C(L)$ at $x \in S \cap \overline{T}$ we see that $S \cap \overline{T}$ is open in S. Moreover, $S \cap \overline{T}$ is also closed in S, since $\overline{T}$ is closed and S is locally closed in X.

Definition 2.3.1.4. *A* **pseudomanifold** *of dimension n is a n-dimensional topologically stratified space X with $X_{n-1} = X_{n-2}$, that is there are no strata of codimension one, and such that $X \smallsetminus X_{n-2}$ is dense in X.*

Pseudomanifolds are a very important class of spaces since Goresky and MacPherson proved in [GM80, GM83] that their intersection (co)homology groups satisfy a generalised Poincaré duality. However, note that Definition 2.3.1.1 is less restrictive than the one of pseudomanifold.

Example 2.3.1.5. *We list some classes of topologically stratified spaces:*

i) Every topological manifold X can be trivially stratified with a single stratum $S = X$.

ii) Every quasi-projective variety can be made into a topologically stratified space with filtration induced by the set of singularities, see [Whi65, Theorem 19.2]. In this case there are only even dimensional strata.

iii) Every compact smooth manifold M with Morse function $f : M \to \mathbb{R}$ can be made into a (Whitney) stratified space. In particular, there is one (contractible) stratum *for each critical point $p \in M$ consisting of all the points whose upward gradient flow (defined using a Riemannian metric on M with respect to which f is Morse-Smale) limits to p, see [Nic11, Theorem 4.3.1].*

Example 2.3.1.6. *We now give some more particular examples.*

i) Let us consider the first quadrant $X = (\mathbb{R}_{\geq 0})^2$ stratified by one axis and the origin, that is

$$X = X_2 \supset X_1 = \mathbb{R}_{\geq 0} \supset X_0 = \{0\}$$

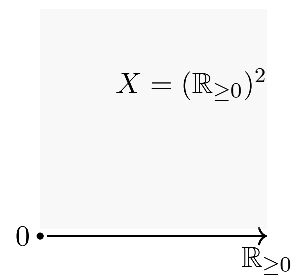

Figure 2.1: $X = (\mathbb{R}_{\geq 0})^2$ stratified by one axis and the origin.

There are three strata, namely

$$\begin{aligned} S_0 &= X_0 \cong \{0\} \\ S_1 &= X_1 \smallsetminus X_0 \cong \mathbb{R}_{>0} \\ S_2 &= X_2 \smallsetminus X_1 \cong \mathbb{R}_{>0} \times \mathbb{R}_{\geq 0}. \end{aligned}$$

In this case, the links are

$$L_0 \cong [0,1] \supset \{0\}, \qquad L_1 \cong \{\mathrm{pt}\}, \qquad L_2 \cong \emptyset.$$

ii) Let us consider the complex line $X = \mathbb{C}$ stratified by a point, which might be taken to be the origin, and its open complement, that is

$$X_2 = \mathbb{C} \supset X_1 = X_0 = \{0\}.$$

In this case, we have two strata given by

$$S_0 = \{0\} \qquad \textit{and} \qquad S_1 = X_2 \smallsetminus X_1 = \mathbb{C} \smallsetminus \{0\},$$

since $X_1 \smallsetminus X_0 = \emptyset$*, with complementary inclusions*

$$S_1 \overset{j}{\hookrightarrow} X \overset{i}{\hookleftarrow} S_0,$$

and links

$$L_0 \cong S^1 \qquad \textit{and} \qquad L_1 \cong \emptyset.$$

iii) As a special case of Example 2.3.1.5.iii), we can consider the n*-dimensional complex projective space and take the Morse function*

$$\begin{aligned} f : \mathbb{P}^n &\longrightarrow \mathbb{R} \\ [z_0, \ldots, z_n] &\mapsto |z_0|^2 + 2|z_1|^2 + \ldots + (n+1)|z_n|^2. \end{aligned}$$

We have a stratification arising from the affine filtration

$$X_{2n} = \mathbb{P}^n \supset X_{2n-1} = X_{2n-2} = \mathbb{P}^{n-1} \supset \ldots \supset X_2 = X_1 = \mathbb{P}^1 \supset X_0 = \mathbb{P}^0 \supset \emptyset$$

with induced by projecting the flag $\mathbb{C}^{n+1} \supset \mathbb{C}^n \supset \ldots \supset \mathbb{C}^1 \supset \mathbb{C}^0$ *of linear subspaces. In this case the strata are given by*

$$S_i = \mathbb{P}^i \smallsetminus \mathbb{P}^{i-1} \cong \mathbb{C}^i \ \forall i = 0, \ldots, n$$

while the links are

$$L_i = S^{2(n-i)-1} \ \forall i = 0, \ldots, n.$$

The first interesting case is given by the complex projective line $X = \mathbb{P}^1$ *stratified by a point and its open complement, that is*

$$X_2 = \mathbb{P}^1 \supset Z = \{\infty\},$$

where $Z = X_1 = X_0$*. In this case we have two strata*

$$S_0 \cong Z \cong \{\infty\} \qquad \textit{and} \qquad S_1 \cong U \cong X \smallsetminus Z \cong \mathbb{C}.$$

Note that the stratum S_0 is closed while S_1 is open. Therefore, we have complementary inclusions denoted by

$$U \overset{j}{\hookrightarrow} X \overset{i}{\hookleftarrow} Z.$$

The links are respectively

$$L_0 \cong S^1 \qquad \text{and} \qquad L_1 \cong \emptyset.$$

iv) There is a real version of iii) above, that is we can consider the n-dimensional real projective space and take the Morse function

$$\begin{aligned} f : \mathbb{RP}^n &\longrightarrow \mathbb{R} \\ [x_0, \ldots, x_n] &\mapsto x_0^2 + 2x_1^2 + \ldots + (n+1)x_n^2. \end{aligned}$$

We have a stratification arising from the affine filtration

$$X_n = \mathbb{RP}^n \supset X_{n-1} = \mathbb{RP}^{n-1} \supset \ldots \supset X_1 = \mathbb{RP}^1 \supset X_0 = \mathbb{RP}^0 \supset \emptyset$$

induced by projecting the flag $\mathbb{R}^{n+1} \supset \mathbb{R}^n \supset \ldots \supset \mathbb{R}^1 \supset \mathbb{R}^0$ of linear subspaces. The first interesting case is given by the circle $X \cong S^1$ stratified by a point and its open complement, that is

$$X_1 = S^1 \supset X_0 = \{0\}.$$

There are two strata

$$S_0 \cong \{0\} \qquad \text{and} \qquad S_1 \cong S^1 \setminus \{0\} \cong (-1, 1)$$

that is

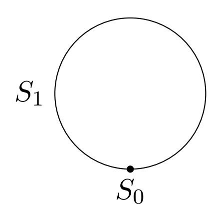

Figure 2.2: Strata of $X = S^1$ stratified by a point and its complement.

The links are

$$L_0 \cong \{-1, 1\} \qquad \text{and} \qquad L_1 = \emptyset.$$

v) Let us consider the torus $X = T^2$ together with a Morse function $f : X \to \mathbb{R}$ and the downwards flow $V = -\nabla f$ with respect to a generic Riemannian metric. There are four critical points that we label by $2, 1, 1', 0$ such that

$$\mathrm{ind}_f(2) = 2, \quad \mathrm{ind}_f(1) = \mathrm{ind}_f(1') = 1 \quad \mathrm{ind}_f(0) = 0.$$

There is a corresponding cell decomposition given by considering the descending manifold D_p at a critical point $p \in X$. In this case we have

$$D_0 \cong \{\mathrm{pt}\}, \; D_1 \cong D_{1'} \cong \mathbb{R}, \; D_2 \cong \mathbb{R}^2.$$

This cell decomposition is in fact a Whitney stratification with strata D_p. We can represent this stratification by seeing the torus as the quotient $T^2 \cong \mathbb{R}^2/\mathbb{Z}^2$, that is

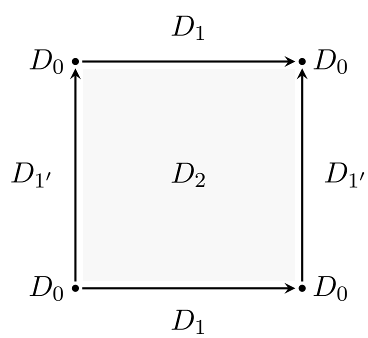

Figure 2.3: Stratification of $X = T^2$.

In this situation, the links are

$$L_0 \cong \{\mathrm{pt}\} \sqcup \{\mathrm{pt}\} \sqcup \{\mathrm{pt}\} \sqcup \{\mathrm{pt}\} \quad L_1 \cong L_{1'} \cong \mathbb{R} \sqcup \mathbb{R} \quad L_0 \cong \emptyset.$$

We now introduce some definitions regarding maps between topologically stratified spaces.

Definition 2.3.1.7. *A continuous map $f : X \to Y$ between two topologically stratified* *spaces is* **stratum preserving** *if for each stratum S of Y the inverse image $f^{-1}(S)$ is a union of strata of X.*

Definition 2.3.1.8. *We say that a map $f : X \to Y$ between two topologically stratified spaces is a* **stratified map** *if it is a stratum preserving map such that for each stratum S of Y the restriction $f : f^{-1}(S) \to S$ is a locally trivial fibre bundle with fibres topologically stratified spaces, that is for each $y \in S$ there exist a neighbourhood N_y of y in S, a topologically stratified space F_y and a stratum preserving homeomorphism*

$$\phi_y : N_y \times F_y \to f^{-1}(N_y)$$

where $N_y \times F_y$ has the product stratification.

2.3.2 Constructible Derived Category

In this part, we introduce the constructible derived category of a topologically stratified space. The main references for this part are [Dim04, KS13].
Let X be a topological space and R a commutative ring, we denote by $\mathbf{Sh}(X)$ the abelian category of R-modules on X and by $\mathrm{C}(X) = \mathrm{C}(\mathbf{Sh}(X))$ the category of chain complexes of sheaves on X. The abelian category $\mathbf{Sh}(X)$ is stable under the functors $f_*, f^*, f_!$. In the special case of a closed inclusion $i : Z \hookrightarrow X$ we have that $i_* = i_!$ is the extension by zero while i^* is the stalk functor. On the other hand, for an open inclusion $j : U \hookrightarrow X$ we have that j_* is the direct image, $j_!$ is the extension by zero and j^* is the restriction functor. The fact that category $\mathbf{Sh}(X)$ contains a lot of information about the space X makes its study very complicated. Therefore, one tries to study a simpler category, namely the category of local systems, which still retains a lot of information about X.

Definition 2.3.2.1. *A* **local system** *$\mathcal{L}$ on X is a locally constant sheaf on X, that is there exist an open covering $\{U_i\}$ of X and a family of R-modules $(F_i) \in R$-***mod** *such that $\mathcal{L}|_{U_i} \cong \mathcal{F}_i$, the constant sheaf on U_i associated to F_i, that is the stalks are finitely generated R-modules.*

We denote by $\mathbf{Loc}(X)$ the abelian category of R-local systems on X. We have the following characterisation of the category of local systems.

Proposition 2.3.2.2. *[Dim04, Proposition 2.5.1] Let X be a paracompact, Hausdorff, path-connected and locally one-connected topological space. The following categories are equivalent:*

i) $\mathbf{Loc}(X)$*.*

ii) Covariant functors from the fundamental grupoid of X to R-modules.

iii) $\pi_1(X, x_0)$**-mod**.

In this thesis we will mainly deal with local systems with coefficients in an algebraically closed field $\Bbbk$, therefore, if not otherwise specified, we will assume so. The study of the category $\mathbf{Loc}(X)$ is more manageable than $\mathbf{Sh}(X)$, but it still captures the information on $\pi_1(X)$ up to Morita equivalence. Moreover, $\mathbf{Loc}(X)$ behaves well under the duality which associates to a local system $\mathcal{L} \in \mathbf{Loc}(S)$ its dual $\mathcal{L}^\vee = \mathrm{Hom}(\mathcal{L}, \Bbbk)$. Furthermore, the functor f^* is well-defined on $\mathbf{Loc}(X)$. On the other hand, in the passage from $\mathbf{Sh}(X)$ to $\mathbf{Loc}(X)$ we lose the fact that f_* is well defined, as for instance it is easy to see that the direct image of the skyscraper sheaf is very far from being a local system, see [Dim04, Exercise 2.5.2.ii)].

We now introduce the notion of stratification, which allows to decompose a space in a union of strata, as we want to further investigate constructible sheaves on a topologically stratified space.

Definition 2.3.2.3. *A finite decomposition $\mathscr{S}$ of X into non-empty disjoint locally closed subsets, called* **strata***, is called a* **stratification** *if the closure of any stratum is a union of strata.*

Let $\mathscr{S}$ be a stratification of X, we denote by $i_S : S \hookrightarrow X$ the inclusion of a stratum into X.

Definition 2.3.2.4. *A sheaf $\mathcal{F} \in \mathbf{Sh}(X)$ is said to be* **constructible** *with respect to $\mathscr{S}$ if for any $S \in \mathscr{S}$ the sheaf $i_S^*\mathcal{F}$ is locally constant.*

We denote by $\mathbf{Constr}(X)$ the category of constructible sheaves on X. The abelian category of constructible sheaves on X turns out to be stable under the functors $f_*, f^*, f_!$, see [Dim04, Theorem 4.1.5]. Moreover, $\mathbf{Constr}(X)$ contains the information about $\pi_1(S)$ for any stratum S of X up to Morita equivalence as well as $\pi_0(L_{S\subset T})$, where $L_{S\subset T} := L_S \cap T$ denotes the link of S in T. Hence, in the step from local systems to constructible sheaves we gain more information about X, but we still do not have a complete formalism of functors, since for instance $f_!$ does not have a well-defined adjoint. In order to do so, we need to generalise further the previous situation.

Definition 2.3.2.5. *A bounded complex of sheaves $\mathcal{F}^\bullet \in \mathbf{D}(X)$ is said to be* **cohomologically constructible** *with respect to $\mathscr{S}$ if all the sheaves $H^i(\mathcal{F}^\bullet)$ are constructible with finitely generated fibres (over R).*

We denote by $\mathbf{D}_{\mathrm{c}}(X)$ the **constructible derived category of** X which is the full triangulated subcategory of the bounded derived category $\mathbf{D}(X)$ consisting of constructible complexes. The first thing to note is that, the category $\mathbf{D}_{\mathrm{c}}(X)$ is not abelian anymore, but only triangulated. On the other hand, if we denote by

$$U \overset{j}{\hookrightarrow} X \overset{i}{\hookleftarrow} Z$$

the complementary inclusions of an open and closed union of strata of X respectively, we have exact functors of triangulated categories, see [BBD82, Section 1.4.1] and [Dim04, Proposition 5.2.2],

$$\mathbf{D}_{\mathrm{c}}(Z) \xrightarrow{i_*} \mathbf{D}_{\mathrm{c}}(X) \xrightarrow{j^*} \mathbf{D}_{\mathrm{c}}(U)$$

such that:

- i_* has i^* and $i^!$ as left and right adjoint respectively.
- j^* has $j_!$ and j_* as left and right adjoint respectively
- We have
  $$j^* i_* = i^* j_! = i^! j_* = 0$$
  and
  $$\mathrm{Hom}_{\mathbf{D}_{\mathrm{c}}(X)}(j_!\mathcal{B}, i_*\mathcal{A}) = \mathrm{Hom}_{\mathbf{D}_{\mathrm{c}}(X)}(i_*\mathcal{A}, j_*\mathcal{B}) = 0$$
  for any $\mathcal{A} \in \mathbf{D}_{\mathrm{c}}(Z)$ and $\mathcal{B} \in \mathbf{D}_{\mathrm{c}}(U)$.
- For any $\mathcal{F} \in \mathbf{D}_{\mathrm{c}}(X)$ there exist unique morphisms $d : j_*j^*\mathcal{F} \to i_*i^!\mathcal{F}[1]$ and $d' : i_*i^*\mathcal{F} \to j_!j^*\mathcal{F}[1]$ such that
  $$i_*i^!\mathcal{F} \to \mathcal{F} \to j_*j^*\mathcal{F} \overset{d}{\to} i_*i^!\mathcal{F}[1]$$
  $$j_!j^*\mathcal{F} \to \mathcal{F} \to i_*i^*\mathcal{F} \overset{d'}{\to} j_!j^*\mathcal{F}[1]$$
  are distinguished triangles.

- The adjunction morphisms

$$i^*i_* \to \mathrm{id} \to i^!i_*$$
$$j^*j_* \to \mathrm{id} \to j^*j_!$$

are isomorphisms, that is the functors i_*, j_* and $j_!$ are fully faithful.

Therefore, in the context of the constructible derived category of X we have the so called six functors formalism, which we can schematise in the following diagram

$$\begin{tikzcd} \mathbf{D}_c(Z) \arrow[r, "i_*"] & \mathbf{D}_c(X) \arrow[l, bend right, "i^*"'] \arrow[l, bend left, "i^!"] \arrow[r, "j^*"] & \mathbf{D}_c(U) \arrow[l, bend right, "j_!"'] \arrow[l, bend left, "j_*"] \end{tikzcd} . \tag{2.3}$$

Remark 2.3.2.6. *Recall that in order to produce a functor at the level of the derived category from a non-exact functor F of constructible sheaves, one has to consider its derived functor, usually denotes by RF (or LF). For the sake of simplicity, throughout this thesis we dropped the R (or L) in front of the symbol of the functor. For example in (2.3) we write j_* in place of Rj_* and so on. Moreover, we adopt the convention used in most of the literature and suppress the extension by zero functor, making clear which category we are working in.*

The category $\mathbf{D}_c(X)$ has the Poincaré-Verdier duality $\mathcal{D}$, which is an involution, that is $\mathcal{D}^2 \cong \mathrm{id}$. Such duality, together with the six functor formalism, gives to $\mathbf{D}_c(X)$ a very powerful framework. Moreover, $\mathbf{D}_c(X)$ encodes all the information about the cohomology of the strata and links, that is it contains information about $H^*(S)$ for any stratum $S \subset X$ and $H^*(L_{S\subset T})$. Moreover, morphisms in the constructible derived category are (generalised) cohomological classes. The downside of this setting is that we are now working in a triangulated category, hence we have lost the advantage of working in an abelian category. We would like to consider some abelian subcategories of $\mathbf{D}_c(X)$ which have the six functor formalism and the duality. In other words, we would like to extend the special case $\mathbf{Constr}(X) \subset \mathbf{D}_c(X)$ to better behaved abelian subcategories. We will do that by considering the heart of a perverse t-structure on $\mathbf{D}_c(X)$ which depends on the choice of a perversity function. In doing that, we will recover the constructible sheaves as the heart of the zero perverse t-structure on the constructible derived category.

In the above setting, the next result gives a useful relation between i^*j_* and $i^!j_!$ in the constructible derived category.

Lemma 2.3.2.7. *Let X be a topologically stratified space and let $U \overset{j}{\hookrightarrow} X \overset{i}{\hookleftarrow} Z$ be complementary inclusions of an open and closed union of strata of X respectively. Then, in $\mathbf{D}_c(X)$, we have $i^*j_* \cong i^!j_![1]$.*

Proof. Let us consider $\mathcal{E} \in \mathbf{D}_{\mathrm{c}}(X)$ and let $\mathcal{F} \cong j^*\mathcal{E} \in \mathbf{D}_{\mathrm{c}}(U)$. We have two triangles

$$j_!\mathcal{F} \to \mathcal{E} \to i^*\mathcal{E} \to j_!\mathcal{F}[1]$$
$$i^!\mathcal{E} \to \mathcal{E} \to j_*\mathcal{F} \to i^!\mathcal{E}[1].$$

By applying the functor $i^!$ to the first triangle and i^* to the latter, we obtain respectively

$$i^!j_!\mathcal{F} \to i^!\mathcal{E} \to i^*\mathcal{E} \to i^!j_!\mathcal{F}[1]$$
$$i^!\mathcal{E} \to i^*\mathcal{E} \to i^*j_*\mathcal{F} \to i^!\mathcal{E}[1].$$

Rotating one of the two triangles above and using the triangulated Five Lemma, see [HJR10, Proposition 4.3], gives $i^*j_*\mathcal{F} \cong i^!j_!\mathcal{F}[1]$. □

Let us consider $\mathcal{E} \in \mathbf{D}_{\mathrm{c}}(X)$. The following Lemma relates the cohomology stalks of $i^*j_*j^*\mathcal{E}$ at a point x in a closed stratum S of X and the cohomology of the link of S with coefficients in the restriction of $\mathcal{E}$ to the link L. We give a detailed explanation as we could not find a reference in the literature.

Lemma 2.3.2.8. *Let X be a topologically stratified space. Let $i : S \hookrightarrow X$ be the inclusion of a closed stratum into X with complementary open map $j : X \setminus S \hookrightarrow X$ and $\mathcal{E} \in \mathbf{D}_c(X)$. Then, for $x \in S$ the cohomology stalks*

$$H^k_x(i^*j_*j^*\mathcal{E}) \cong H^k(L; \mathcal{E}|_L) \quad \forall k \in \mathbb{Z},$$

where L is (any choice of) a link of S at x and $\mathcal{E}|_L \in \mathbf{D}_c(L)$ is constructed below (up to isomorphism). Moreover, if $\pi_1(S)$ is finite, then

$$i^*j_*j^*\mathcal{E} \cong \bigoplus_{k \in \mathbb{Z}} \mathcal{L}_k[-k],$$

where $\mathcal{L}_k \in \mathbf{Loc}(S)$ is a local system with stalks $H^k(L; \mathcal{E}|_L)$.

Proof. Recall that x has an open neighbourhood V with stratum preserving homeomorphism

$$V \cong \mathbb{R}^d \times C(L),$$

where $d = \dim(S)$ and $C(L) = L \times [0,1)/L \times \{0\}$ is the open cone on the link L, which is a compact topologically stratified space of dimension $\operatorname{codim}(S) - 1$. This homeomorphism maps $V \cap S$ to $\mathbb{R}^d \times V$ and x to $(0, v)$, where v is the vertex of the cone $C(L)$. Let

$$\pi : V \smallsetminus S \cong \mathbb{R}^d \times (C(L) \smallsetminus v) \cong \mathbb{R}^d \times (0,1) \times L \to L$$

be the projection and $\sigma : L \hookrightarrow V \smallsetminus S$ any section. Since $\mathcal{E}|_{V \smallsetminus S}$ is cohomologically constant on the fibres of π, which are contractible, the counit

$$\pi^*\pi_*(\mathcal{E}|_{V \smallsetminus S}) \to \mathcal{E}|_{V \smallsetminus S}$$

of the adjunction is an isomorphism in $\mathbf{D}_{\mathrm{c}}(V \smallsetminus S)$, see [GM83, 1.13.(17)]. Since $\pi\sigma = \mathrm{id}$, it follows that

$$\begin{aligned} \pi_*(\mathcal{E}|_{V \smallsetminus S}) &\cong \sigma^*\pi^*\pi_*(\mathcal{E}|_{V \smallsetminus S}) \\ &\cong \sigma^*(\mathcal{E}|_{V \smallsetminus S}) \end{aligned}$$

in $\mathbf{D}_{\mathrm{c}}(L)$. We define $\mathcal{E}|_L$ to be (any choice in the isomorphism class of) this object. Note that $\mathcal{E}|_L$ is independent of the choice of section σ and that

$$\begin{aligned} H^k(L; \mathcal{E}_L) &\cong H^k(L; \pi_*(\mathcal{E}|_{V \smallsetminus S})) \\ &\cong H^k(V \smallsetminus S; \mathcal{E}|_{V \smallsetminus S}) \\ &\cong H^k(V; j_*j^*\mathcal{E}|_V). \end{aligned}$$

Since x has a cofinal sequence of neighbourhoods of the form V, it follows that

$$H^k_x(i^*j_*j^*\mathcal{E}) \cong H^k(L; \mathcal{E}|_L)$$

for all $k \in \mathbb{Z}$ as claimed. In particular, the right hand side is well-defined up to isomorphism. If S has finite fundamental group, then $\mathbf{D}_{\mathrm{c}}(S)$ is semisimple, see Remark 3.1.0.5 and so

$$i^*j_*j^*\mathcal{E} \cong \bigoplus_{k \in \mathbb{Z}} H^k(i^*j_*j^*\mathcal{E})[-k]$$

is a direct sum of local systems with the stated stalk. □

Remark 2.3.2.9. *The subtle point in Lemma 2.3.2.8 is that whilst the open cone $c(L)$ is unique up to isomorphism, the link L itself is not. However, the groups $H^k(L; \mathcal{E}|_L)$ are independent of the choice of L (up to isomorphism).*

2.3.3 Perversities

We now introduce perversity functions on a topologically stratified space X. They will play a crucial role in the definition of the perverse t-structure on $\mathbf{D}_c(X)$ whose heart will be the category of p-perverse sheaves.

Definition 2.3.3.1. *A* **perversity** *p on a topologically stratified space X is a function*

$$p : \{\textit{strata of } X\} \to \mathbb{Z}.$$

Remark 2.3.3.2. *Given a topologically stratified space X with strata $S_0, \ldots, S_n$, one can write a perversity on X as a vector*

$$p(S) = (p(S_n), \ldots, p(S_0)).$$

Definition 2.3.3.3. *Let X be a topologically stratified space and p a perversity on it. For any stratum S of X, the* **dual perversity** *p^* is defined as*

$$p^*(S) = -\dim_{\mathbb{R}}(S) - p(S).$$

Example 2.3.3.4. *There are some perversities that play a prominent role. Let X be a topologically stratified space and S a stratum of X, then:*

i) The zero perversity is $o(S) = 0$.

ii) The lower middle perversity is $m(S) = \left\lfloor -\frac{\dim(S)}{2} \right\rfloor$.

iiI) The upper middle perversity is $n(S) = \left\lceil -\frac{\dim(S)}{2} \right\rceil$.

iv) The top perversity is defined is $t(S) = -\dim(S)$.

Note that zero and top perversity and lower middle and upper middle perversity are pairs of dual perversities.

Remark 2.3.3.5. *If X is a topologically stratified space with all even dimensional strata, for example a complex algebraic variety, the lower and upper middle perversity coincide and we use the term middle perversity. Moreover, note that the middle perversity is self-dual.*

Remark 2.3.3.6. *Definition 2.3.3.1 allows a perversity to be a vector of numbers with no restrictions. On the other hand, perversities p on a topologically stratified space X such that*

$$t(S) \leq p(S) \leq o(S)$$

for any stratum S of X, play a crucial role since they are more closely related to the geometry of the space X. We refer to them as **geometric perversities**. *On the other hand, the perversities in [Dim04, Section 5.1] are defined only for spaces with even dimensional strata, but they agree with what we call geometric perversities.*

Remark 2.3.3.7. *Our definition of perversity is more general than the one of [GM80], where the authors define a perversity as a function*

$$p : \mathbb{Z}_{\geq 2} \to \mathbb{N}$$

such that

$$\begin{gathered} p(2) = 0 \\ p(i) \leq p(i+1) \leq p(i) + 1 \quad i > 2. \end{gathered}$$

We will refer to such perversities, which are allowed to grow in a controlled way, as Goresky-MacPherson perversity (or GM-perversities for short).

We now show that if p is a geometric perversity on X and both p and p^* are decreasing function depending only on the dimension, then p is a GM-perversity.

Lemma 2.3.3.8. *Let p be a geometric perversity on a topologically stratified space X such that p and its dual p^* are decreasing functions which depends only on the dimension. Then, p and p^* can vary at most by -1 at each step and therefore p is a GM-perversity.*

Proof. The first condition implies that $-\dim(S) \leq p(S) \leq 0$ for any stratum $S \subset X$. Suppose that $\dim(S) \leq \dim(T)$, where S and T are strata of X. Since both p and p^* are decreasing, we have

$$p(S) - p(T) \geq 0 \qquad \text{and} \qquad p^*(S) - p^*(T) \geq 0.$$

Note that the right hand side is equivalent to

$$-\dim(S) - p(S) + \dim(T) + p(T) \geq 0,$$

therefore we have

$$0 \leq p(S) - p(T) \leq \dim(T) - \dim(S).$$

In particular, if $\dim(T) = \dim(S) + 1$, then $0 \leq p(S) - p(T) \leq 1$ and p is a GM-perversity.

□

2.3.4 Perverse Sheaves

In this section, we introduce the abelian category ${}^p\mathbf{Perv}(X)$ of p-perverse sheaves on a topologically stratified space as the heart of the p-perverse t-structure on the constructible derived category of X.

Let X be a topologically stratified space and p a perversity on it. We can consider the following subcategories of $\mathbf{D}_\mathrm{c}(X)$

$$\begin{aligned} {}^p\mathrm{D}^{\leq 0} &= \{\mathcal{E} \in \mathbf{D}_\mathrm{c}(X) \mid H^n(i_S^*\mathcal{E}) = 0 \text{ if } n > p(S)\} \\ {}^p\mathrm{D}^{\geq 0} &= \{\mathcal{E} \in \mathbf{D}_\mathrm{c}(X) \mid H^n(i_S^!\mathcal{E}) = 0 \text{ if } n < p(S)\} \end{aligned} \tag{2.4}$$

where H^n denotes the cohomology sheaf and S is running over all strata of X.

Proposition 2.3.4.1. *[BBD82, 2.1.4],[Dim04, 5.1.19] The pair of subcategories $({}^p\mathrm{D}^{\leq 0}, {}^p\mathrm{D}^{\geq 0})$ is a bounded t-structure on $\mathbf{D}_c(X)$.*

Definition 2.3.4.2. *Let X be a topologically stratified space and p a perversity on it. The category of* p**-perverse sheaves on** X *is the heart of the p-perverse t-structure on $\mathbf{D}_c(X)$, that is*

$${}^p\mathbf{Perv}(X) = {}^p\mathrm{D}^{\leq 0} \cap {}^p\mathrm{D}^{\geq 0}.$$

Remark 2.3.4.3. *It follows from the general theory of t-structures, see [BBD82], that the category of p-perverse sheaves on X is an abelian subcategory of the constructible derived category stable under extensions.*

Remark 2.3.4.4. *The p-perverse t-structure (2.4) on $\mathbf{D}_c(X)$ can be obtained using (inductively) the six functor formalism by glueing p-perverse t-structures on $\mathbf{D}_c(S)$ for strata*

$S \subset X$ given by

$$
\begin{aligned}
{}^p\mathrm{D}^{\leq 0}_S &= \{\mathcal{E} \in \mathbf{D}_c(S) \mid H^n(i_S^*\mathcal{E}) = 0 \text{ if } n > p(S)\} \\
{}^p\mathrm{D}^{\geq 0}_S &= \{\mathcal{E} \in \mathbf{D}_c(S) \mid H^n(i_S^!\mathcal{E}) = 0 \text{ if } n < p(S)\}
\end{aligned}
$$

where $i_S : S \hookrightarrow X$ is the inclusion of a stratum into X, see [BBD82, 1.4.10, 1.4.12, 2.1.8]. Note that the heart of the perverse t-structures on $\mathbf{D}_c(X)$ are the categories of shifted local systems $\mathbf{Loc}(S)[-p(S)]$, that is the category of perverse sheaves is the result of glueing the categories of local systems on the strata with a shift given by the value of perversity on each stratum.

When X has a single stratum S, Verdier duality preserves local systems up to a shift. In particular one can consider a local system $\mathcal{L} \in \mathbf{Loc}(S)$ as a complex concentrated in degree zero in $\mathbf{D}_\mathrm{c}(S)$. Then

$$\mathcal{D}\mathcal{L} = \mathcal{L}^\vee[\dim(S)]$$

where $\mathcal{L}^\vee = \mathrm{Hom}(\mathcal{L}, \Bbbk)$ is the dual local system. Therefore, in view of Remark 2.3.4.4 we have the following result.

Theorem 2.3.4.5. *[BBD82, 2.1.16] Verdier duality on $\mathbf{D}_c(X)$ restricts to*

$$\mathcal{D} : {}^p\mathbf{Perv}(X) \to {}^{p^*}\mathbf{Perv}(X).$$

By taking the hearts of perverse t-structures on $\mathbf{D}_\mathrm{c}(X)$ for different perversities we are then considering different abelian subcategories of $\mathbf{D}_\mathrm{c}(X)$. Moreover, the above result says that Verdier duality sends p-perverse sheaves to p^*-perverse sheaves. We now explain how the six functor formalism on $\mathbf{D}_\mathrm{c}(X)$ descends to the abelian categories of perverse sheaves.

2.3.5 Functors in ${}^p\mathbf{Perv}(X)$

In this section we define perverse functors, see [BBD82, 1.4.15]. We assume that p is a perversity in the sense of Definition 2.3.3.1 on a topologically stratified space X. In parallel to Proposition 2.1.4.8, we have a cohomological functor

$${}^pH^k : \mathbf{D}_\mathrm{c}(X) \to {}^p\mathbf{Perv}(X)$$

called **perverse cohomology** defined as

$$^pH^k := {}^p\tau_{\leq k}{}^p\tau_{\geq k}$$

where ${}^p\tau_{\leq k}$ and ${}^p\tau_{\geq k}$, called p-**perverse truncations**, are the right and left adjoint to the inclusions of ${}^p\mathrm{D}^{\leq n} = {}^p\mathrm{D}^{\leq 0}[-n]$ and ${}^p\mathrm{D}^{\geq n} = {}^p\mathrm{D}^{\geq 0}[-n]$ in $\mathbf{D}_{\mathrm{c}}(X)$ respectively, see [BBD82, 1.1.16]. Let us denote by U an open union of strata of X and $Z = X \smallsetminus U$ its closed complement, that is we have complementary inclusions

$$U \overset{j}{\hookrightarrow} X \overset{i}{\hookleftarrow} Z.$$

Let $\epsilon : {}^p\mathbf{Perv}(F) \hookrightarrow \mathbf{D}_{\mathrm{c}}(F)$ be the inclusion of the heart of a perverse t-structure into the constructible derived category where $F \in \{U, X, Z\}$. For $T \in \{j_!, j_*, j^*, i_*, i^!, i^*\}$, we can define p-**perverse functors** as

$$^pT := {}^pH^0 \circ T \circ \epsilon. \tag{2.5}$$

In particular, the six formalism functor on $\mathbf{D}_{\mathrm{c}}(X)$ described in (2.3) descends to p-perverse sheaves, see [BBD82, 2.1.7]. Therefore, we have the following adjunctions of functors:

$$\begin{tikzcd} {}^p\mathbf{Perv}(Z) \arrow[r, "i_*"] \arrow[r, phantom, shift right=3, "\perp"] & {}^p\mathbf{Perv}(X) \arrow[l, bend right=50, "{}^pi^*"'] \arrow[l, bend right=50, phantom, shift left=2, "\perp"] \arrow[l, bend left=50, "{}^pi^!"] \arrow[r, "j^*"] \arrow[r, phantom, shift right=3, "\perp"] & {}^p\mathbf{Perv}(U) \arrow[l, bend right=50, "{}^pj_!"'] \arrow[l, bend right=50, phantom, shift left=2, "\perp"] \arrow[l, bend left=50, "{}^pj_*"] \end{tikzcd}\ . \tag{2.6}$$

Note that i_* and j^* are exact functors with ${}^pi^*$ and ${}^pj_!$ as left adjoint (hence right exact) and ${}^pi^!$ and ${}^pj_*$ as right adjoint (hence left exact) respectively. Moreover, we have that

$${}^pi^*{}^pj_! = {}^pi^!{}^pj_* = 0,$$

and for any $\mathcal{A} \in {}^p\mathbf{Perv}(Z)$ and $\mathcal{B} \in {}^p\mathbf{Perv}(U)$

$$\mathrm{Hom}_{{}^p\mathbf{Perv}(X)}({}^pj_!\mathcal{B}, i_*\mathcal{A}) = \mathrm{Hom}_{{}^p\mathbf{Perv}(X)}(i_*\mathcal{A}, {}^pj_*\mathcal{B}) = 0$$

Finally, the adjunction morphisms

$$ {}^p i^* i_* \to \mathrm{id} \to {}^p i^* i_* $$
$$ j^{*p} j_* \to \mathrm{id} \to j^{*p} j_! $$

are isomorphisms.
In some cases, perverse functors can be described in a more familiar way.

Remark 2.3.5.1. *Let $f : X \to Y$ be a quasi-finite, that is with finite fibres, and affine morphism of algebraic varieties. Then the functors*

$$ f_*, f_! : \mathbf{D}_c(X) \to \mathbf{D}_c(Y) $$

are t-exact, see [Dim04, Corollary 5.2.17]. Therefore, their perverse version for the middle perversity coincides with the functors at the level of constructible derived categories, that is

$$ {}^m f_* = f_* \quad \text{and} \quad {}^m f_! = f_!. $$

Moreover, in [dCM09, Section 5.3], one can find a list of special cases for f_* and $f_!$ and further observations about (perverse) t-exactness.

Definition 2.3.5.2. *An* **extension** *of $\mathcal{G} \in \mathbf{D}_c(U)$ is an object $\mathcal{F} \in \mathbf{D}_c(X)$ such that $j^*\mathcal{F} \cong \mathcal{G}$.*

Remark 2.3.5.3. *Let $\mathcal{F} \in \mathbf{D}_c(X)$ be an extension of $\mathcal{G} \in \mathbf{D}_c(U)$. Then the isomorphism $j^*\mathcal{G} \cong \mathcal{F}$ gives, by adjunction, morphisms*

$$ j_!\mathcal{G} \to \mathcal{F} \to j_*\mathcal{G}. $$

The composite is a natural morphism $\alpha : j_! \to j_*$ which induces a natural morphism ${}^p\alpha : {}^p j_! \to {}^p j_*$.

We can now define a very important extension of a perverse sheaf supported on U. This notion will be central in the rest of our work, see [BBD82, 1.4.22].

Definition 2.3.5.4. *The* **intermediate extension functor** *is defined as*

$$ {}^p j_{!*} := \mathrm{im}({}^p\alpha) : {}^p\mathbf{Perv}(U) \to {}^p\mathbf{Perv}(X). $$

Remark 2.3.5.5. *i) Note that the construction of the intermediate extension functor makes sense since it takes place in an abelian category (as perverse sheaves are the heart of the perverse t-structure on the derived constructible category).*

ii) We refer to ${}^p j_{!*}$ as the intermediate extension since, by applying ${}^p H^0$ to the natural morphism α of Remark 2.3.5.3 and using that for a perverse sheaf $\mathcal{F} \in {}^p\mathbf{Perv}(X)$, we have ${}^p H^0(\mathcal{F}) \cong \mathcal{F}$, one gets

$$ {}^p j_! \mathcal{G} \to \mathcal{F} \to {}^p j_* \mathcal{F}. $$

Thus, ${}^p j_!$ and ${}^p j_*$ are initial and final respectively in the category of extensions of $\mathcal{G} \in {}^p\mathbf{Perv}(U)$ as we will further investigate in Chapter 5, in particular Proposition 5.1.3.1. Therefore we have

$$ \begin{tikzcd} {}^p j_! \mathcal{G} \arrow[rr, "\alpha"] \arrow[dr, two heads] & & {}^p j_* \mathcal{G} \\ & {}^p j_{!*} \mathcal{G} \arrow[ur, hook] & \end{tikzcd} . $$

Furthermore, see [BBD82, 1.4.22], the initial extension ${}^p j_! \mathcal{G}$ of $\mathcal{G} \in {}^p\mathbf{Perv}(U)$ is the unique extension $\mathcal{F}$ of $\mathcal{G}$ in $\mathbf{D}_c(X)$ such that $i^*\mathcal{F} \in {}^p\mathrm{D}^{\leq -2}(Z)$ and $i^!\mathcal{F} \in {}^p\mathrm{D}^{\geq 0}(Z)$. Dually, the final extension ${}^p j_* \mathcal{G}$ of $\mathcal{G} \in {}^p\mathbf{Perv}(U)$ is the unique extension $\mathcal{F}'$ of $\mathcal{G}$ in $\mathbf{D}_c(X)$ such that $i^*\mathcal{F}' \in {}^p\mathrm{D}^{\leq 0}(Z)$ and $i^!\mathcal{F}' \in {}^p\mathrm{D}^{\geq 2}(Z)$.

iii) One can show, see [BBD82, 1.4.22 and 1.4.24], that the intermediate extension $\mathcal{F} \cong {}^p j_{!*} \mathcal{G}$ is the unique extension of $\mathcal{G} \in \mathbf{D}_c(U)$ satisfying the **strong conditions**

$$ i^*\mathcal{F} \in {}^p\mathrm{D}^{\leq -1}(Z) \quad \textit{and} \quad i^!\mathcal{F} \in {}^p\mathrm{D}^{\geq 1}(Z). $$

The intermediate extension can be further characterised as follows.

Lemma 2.3.5.6. *[BBD82, Corollary 1.4.25] The intermediate extension ${}^p j_{!*} \mathcal{G}$ of $\mathcal{G} \in {}^p\mathbf{Perv}(U)$ is characterised as the unique extension of $\mathcal{G} \in {}^p\mathbf{Perv}(U)$ with no non-zero sub-object or quotient supported on Z.*

We now discuss the exactness of the functor ${}^p j_{!*}$.

Lemma 2.3.5.7. *The intermediate extension functor ${}^p j_{!*}$ preserves monomorphisms and epimorphisms.*

Proof. Let us consider a short exact sequence

$$0 \to \mathcal{E} \xrightarrow{\alpha} \mathcal{F} \xrightarrow{\beta} \mathcal{G} \to 0$$

in ${}^p\mathbf{Perv}(X)$. Since ${}^pj_!$ and ${}^pj_*$ are respectively right and left exact, see Section 2.3.5, we have a diagram

$$\begin{tikzcd}
 & {}^pj_!\mathcal{E} \arrow[r] \arrow[d, two heads] & {}^pj_!\mathcal{F} \arrow[r] \arrow[d, two heads] & {}^pj_!\mathcal{G} \arrow[r] \arrow[d, two heads] & 0 \\
 & {}^pj_{!*}\mathcal{E} \arrow[r, "{}^pj_{!*}\alpha"] \arrow[d, hook] & {}^pj_{!*}\mathcal{F} \arrow[r, "{}^pj_{!*}\beta"] \arrow[d, hook] & {}^pj_{!*}\mathcal{G} \arrow[d, hook] & \\
0 \arrow[r] & {}^pj_*\mathcal{E} \arrow[r] & {}^pj_*\mathcal{F} \arrow[r] & {}^pj_*\mathcal{G} &
\end{tikzcd}$$

which shows that ${}^pj_{!*}\alpha$ and ${}^pj_{!*}\beta$ are respectively a monomorphism and an epimorphism. □

Remark 2.3.5.8. *Note also that ${}^pj_{!*}$ might be not exact as the condition $\mathrm{im}\,{}^pj_{!*}\alpha = \ker {}^pj_{!*}\beta$ can fail. In [SW18, Example 3.11] the authors give an instance of a two dimensional local system on $\mathbb{C}^*$ for which ${}^pj_{!*}$ is not exact.*

Another important property of the intermediate extension functor is that it preserves simple and indecomposable objects.

Lemma 2.3.5.9. *Let $\mathcal{F} \in {}^p\mathbf{Perv}(U)$.*

i) $\mathcal{F}$ is simple if and only if ${}^pj_{!*}\mathcal{F} \in {}^p\mathbf{Perv}(X)$ is simple.

ii) $\mathcal{F}$ is indecomposable if and only if ${}^pj_{!*}\mathcal{F} \in {}^p\mathbf{Perv}(X)$ is indecomposable.

Proof. For the first statement see [BBD82, Section 4.3]. For the second one, assume that $\mathcal{F}$ is indecomposable and suppose that ${}^pj_{!*}\mathcal{F} \cong \mathcal{G} \oplus \mathcal{H}$. Then $\mathcal{F} \cong j^*\mathcal{G} \oplus j^*\mathcal{H}$ and without loss of generality we can assume $j^*\mathcal{G} \cong 0$. Hence $\mathcal{G}$ is supported on Z, so $\mathcal{G} \cong 0$ by Lemma 2.3.5.6. □

Lemma 2.3.5.10. *Let $j : U \hookrightarrow X$ be the inclusion of an open union of strata. The intermediate extension functor ${}^pj_{!*}$ is fully faithful and the restriction to U induces a monomorphism*

$$\mathrm{Ext}^1_{{}^p\mathbf{Perv}(X)}({}^pj_{!*}\mathcal{E}, {}^pj_{!*}\mathcal{F}) \hookrightarrow \mathrm{Ext}^1_{{}^p\mathbf{Perv}(U)}(\mathcal{E}, \mathcal{F})$$

for any $\mathcal{E}, \mathcal{F} \in {}^p\mathbf{Perv}(U)$. Moreover, the intermediate extension functor ${}^pj_{!*}$ is exact if and only if the above is an isomorphism for all $\mathcal{E}, \mathcal{F} \in {}^p\mathbf{Perv}(X)$.

Proof. Let us consider the long exact sequence induced by applying the functor $\mathrm{Hom}_{{}^p\mathbf{Perv}(X)}({}^pj_{!*}\mathcal{E}, -)$ to the triangle

$$i_*i^!{}^pj_{!*}\mathcal{F} \to {}^pj_{!*}\mathcal{F} \to j_*\mathcal{F} \to i_*i^!{}^pj_{!*}\mathcal{F}[1].$$

Using Remark 2.3.5.5.iii) and the fact that i^* and $i^!$ are respectively right and left t-exact, see Section 2.3.5, we have

$$\mathrm{Hom}_{{}^p\mathbf{Perv}(X)}({}^pj_{!*}\mathcal{E}, i_*i^!{}^pj_{!*}\mathcal{F}[d]) \cong \mathrm{Hom}_{\mathbf{D}_c(X \smallsetminus U)}(i^*{}^pj_{!*}\mathcal{E}, i^!{}^pj_{!*}\mathcal{F}[d]) \cong 0 \quad \text{for } d = 0, 1.$$

Therefore

$$\begin{aligned}\mathrm{Hom}_{{}^p\mathbf{Perv}(X)}({}^pj_{!*}\mathcal{E}, {}^pj_{!*}\mathcal{F}) &\cong \mathrm{Hom}_{\mathbf{D}_c(X)}({}^pj_{!*}\mathcal{E}, j_*\mathcal{F}) \\ &\cong \mathrm{Hom}_{{}^p\mathbf{Perv}(U)}(\mathcal{E}, \mathcal{F}),\end{aligned}$$

that is ${}^pj_{!*}$ is fully faithful. Moreover, there is a long exact sequence

$$0 \to \mathrm{Hom}_{\mathbf{D}_c(X)}({}^pj_{!*}\mathcal{E}, {}^pj_{!*}\mathcal{F}[1]) \to \mathrm{Hom}_{\mathbf{D}_c(X)}(\mathcal{E}, \mathcal{F}[1]) \to \mathrm{Hom}_{\mathbf{D}_c(X)}(i^*{}^pj_{!*}\mathcal{E}, i^!{}^pj_{!*}\mathcal{F}[2]) \to \dots$$

which gives the inclusion of the Ext-groups. If ${}^pj_{!*}$ is exact, it induces an inverse to such inclusion, therefore the latter is an isomorphism. On the other hand, if the inclusion of the Ext-groups in the statement is an isomorphism, let us consider a short exact sequence

$$0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{E} \to 0$$

in ${}^p\mathbf{Perv}(U)$. It lifts, uniquely up to isomorphism of algebraic extensions, to a short exact sequence

$$0 \to {}^pj_{!*}\mathcal{F} \to \mathcal{H} \to {}^pj_{!*}\mathcal{E} \to 0$$

where $j^*\mathcal{H} \cong \mathcal{G}$. The perverse sheaf $\mathcal{H} \in {}^p\mathbf{Perv}(X)$ cannot have non-zero sub-objects or quotients supported on $X \smallsetminus U$, hence $\mathcal{H} \cong {}^pj_{!*}\mathcal{G}$. Since ${}^pj_{!*}$ is fully faithful, the lifted short exact sequence is the one obtained by applying ${}^pj_{!*}$ to the original one. Thus ${}^pj_{!*}$ is exact. □

Let $\mathcal{E} \in {}^p\mathbf{Perv}(U)$, then while ${}^pj_{!*}\mathcal{E} \in {}^p\mathbf{Perv}(X)$ represents the minimal extension of $\mathcal{E}$.

When X is a complex irreducible variety, it is possible to define the **Beilinson's maximal extension** $\mathcal{M} \in {}^p\mathbf{Perv}(X)$, see [Bei87a], where such extension was defined for the first time, and [Rei10], for more details and the definition in terms of (unipotent) nearby and vanishing cycles functors. In Chapter 5 we will show that for a topologically stratified spaces with finitely many strata, each with finite fundamental group, we can define the maximal extension in a way that it agrees with Beilinson's one in the case of complex algebraic varieties.

Moreover, dually to the situation described in Remark 2.3.5.3, given a perverse sheaf $\mathcal{F} \in {}^p\mathbf{Perv}(X)$ there is a natural morphism $\beta : {}^pi^!\mathcal{F} \to {}^pi^*\mathcal{F}$. Therefore, dually to the intermediate extension functor, one can define another important functor.

Definition 2.3.5.11. *Let* $\mathcal{F} \in {}^p\mathbf{Perv}(X)$*, the* **intermediate restriction functor** *is defined as*

$$ {}^pi^{!*}\mathcal{E} := \mathrm{im}({}^pi^!\mathcal{E} \xrightarrow{\beta} {}^pi^*\mathcal{E}) : {}^p\mathbf{Perv}(X) \to {}^p\mathbf{Perv}(Z). $$

We will make use of the intermediate restriction functor in Chapter 5. It will play an important role in the classification of indecomposable perverse sheaves.

2.3.6 Properties of Perverse Sheaves

In this section recall some properties about the category of p-perverse sheaves ${}^p\mathbf{Perv}(X)$, the main reference is [BBD82]. We assume that X is a topologically stratified space, p a perversity on it and $\Bbbk$ is an algebraically closed field.

a) Let $i_S : S \hookrightarrow X$ be the inclusion of a stratum into X. The simple p-perverse sheaves are those of the form

$$ {}^pi_{S!*}\mathcal{L}[-p(S)] $$

where $\mathcal{L} \in \mathbf{Loc}(S)$ is an irreducible local system on a stratum S, see [BBD82, Theorem 4.3.1.ii)].

b) Simple perverse sheaves are intersection cohomology complexes, see [GM83], since their cohomology groups are the p-intersection cohomology groups of the closure $\overline{S}$ with coefficients in $\mathcal{L} \in \mathbf{Loc}(S)$, that is

$$ {}^p\mathcal{IC}(\overline{S}, \mathcal{L}) \cong {}^pi_{S!*}\mathcal{L}[-p(S)], $$

see in particular [GM83, Theorem in Section 3.5] and [BBD82, Proposition 2.3.4].

c) If X is a complex irreducible variety and we consider the middle perversity, see Remark 2.3.3.5, the category of m-perverse sheaves is a faithful heart in the sense of Definition 2.1.5.1, see [Bei87b]. Note that in general the p-perverse t-structure might not be faithful. For example, for $X = S^2$ with a single stratum we have that ${}^p\mathbf{Perv}(X) \cong \mathbf{Loc}(X)[-p(S)] \cong \mathbf{Vect}_{\Bbbk}$ for any perversity, see Example 2.3.7.1. Therefore, at the level of derived category, we have $\mathbf{D}^b({}^p\mathbf{Perv}(X)) \simeq \mathbf{D}^b(\mathbf{Vect}_{\Bbbk})$ and in particular $\mathrm{Ext}^i(\mathcal{E},\mathcal{E}) = 0$ for $i \neq 0$. However, for $\Bbbk_X \in \mathbf{D}_{\mathrm{c}}(X)$ we have $\mathrm{Ext}^2(\Bbbk_X, \Bbbk_X) \cong H^2(X;\Bbbk) \neq 0$.

d) The category ${}^p\mathbf{Perv}(X)$ is artinian and noetherian, see [BBD82, Theorem 4.3.1.i)], hence it is a finite length category. Therefore, every object has a well-defined composition series with simple factors.

e) The category ${}^p\mathbf{Perv}(X)$ is a Krull-Remak-Schmidt category, that is every object has a unique expression (up to isomorphism) as a direct sum of indecomposable objects.

Remark 2.3.6.1. *Note that e) implies that being able to describe the Auslander-Reiten quiver of* ${}^p\mathbf{Perv}(X)$*, see Remark 2.2.3.26, gives a complete characterisation of the category of p-perverse sheaves on X (at least under the more restrictive hypothesis of finite representation type).*

f) A perverse sheaf is not a sheaf, more precisely it is a complex of sheaves. However, there is a functor $U \mapsto {}^p\mathbf{Perv}(U)$ for U an open subset of X which behaves like a sheaf. Thus the category ${}^p\mathbf{Perv}(X)$ is stack, [BBD82, 2.1.23], [KS13, Proposition 10.2.9] and [Dim04, Remark 5.1.17]. In particular, a morphism is the zero morphism in ${}^p\mathbf{Perv}(X)$ if and only if it is the zero morphism when restricted to an open cover. Note that this is not true for $\mathbf{D}_{\mathrm{c}}(X)$.

g) If X has finitely many strata S, each with finite fundamental group, then the category ${}^p\mathbf{Perv}(X)$ has finitely many simple objects. In particular, simple perverse sheaves are in one-to-one correspondence with irreducible local systems on strata S of X, which are simple objects in $\mathbf{Loc}(S)$, and with irreducible representations of $\pi_1(S)$. Indeed, if $j : U \hookrightarrow X$ and $i : Z \hookrightarrow X$ are complementary open and closed unions of strata respectively, then the simple objects in ${}^p\mathbf{Perv}(X)$ are either of the form ${}^p j_{!*}\mathcal{E}$, for $\mathcal{E} \in {}^p\mathbf{Perv}(U)$ simple perverse sheaf on U, or $i_*\mathcal{G}$, for $\mathcal{G} \in {}^p\mathbf{Perv}(Z)$ simple perverse sheaf on Z, see [BBD82, Proposition 1.4.26].

Let S be a closed stratum of X and let denote by $i : S \hookrightarrow X$ the inclusion of S into X. We can give a characterisation of maximal sub-objects and quotients of a p-perverse sheaf in terms of perverse functors.

Lemma 2.3.6.2. *Let X be a topologically stratified space and $\mathcal{E} \in {}^p\mathbf{Perv}(X)$.*

i) $i_* {}^p i^! \mathcal{E} \hookrightarrow \mathcal{E}$ is the inclusion of the maximal sub-object supported on S.

ii) $\mathcal{E} \twoheadrightarrow i_* {}^p i^* \mathcal{E}$ is the projection on the maximal quotient supported on S.

Proof. Note that the two statements are dual to each other. Therefore we prove only the first one. Let us consider the triangle

$$i_* i^! \mathcal{E} \to \mathcal{E} \to j_* j^* \mathcal{E} \to i_* i^! \mathcal{E}[1]$$

where $j : U = X \smallsetminus S \hookrightarrow X$ is the open inclusion of the complement of S into X. By taking perverse cohomology, we have a long exact sequence of the form

$$\ldots \to {}^pH^{-1}(j_* j^* \mathcal{E}) \to i_* {}^p i^! \mathcal{E} \to \mathcal{E} \to {}^p j_* j^* \mathcal{E} \to i_* {}^pH^1(i^! \mathcal{E}) \to {}^pH^1(\mathcal{E}) \to \ldots$$

Since $\mathcal{E} \in {}^p\mathbf{Perv}(X)$ and j_* is left exact, the above reduces to

$$0 \to i_* {}^p i^! \mathcal{E} \to \mathcal{E} \to {}^p j_* j^* \mathcal{E} \to i_* {}^pH^1(i^! \mathcal{E}) \to \ldots$$

giving the inclusion $i_* {}^p i^! \hookrightarrow \mathcal{E}$. Now, let us suppose that there is another sub-object of $\mathcal{E}$ supported on S, that this $\mathcal{E}' \hookrightarrow \mathcal{E}$. Then, we have a diagram

$$\begin{tikzcd} {}^p i^! \mathcal{E}' \arrow[r] \arrow[d, "\cong"'] & {}^p i^! \mathcal{E} \arrow[d, hook] \\ \mathcal{E}' \arrow[r, hook] & \mathcal{E} \end{tikzcd}$$

which shows that ${}^p i^! \mathcal{E}' \hookrightarrow {}^p i^! \mathcal{E}$. Therefore ${}^p i^! \mathcal{E}$ is maximal. $\square$

Using Lemma 2.3.2.8, we give a characterisation of the restriction of a p-perverse sheaf to (a) link. We then deduce a vanishing result for the cohomology of the link.

Lemma 2.3.6.3. *Let S be a stratum of X, let L be a link of S and $\mathcal{F} \in {}^p\mathbf{Perv}(X)$. Then, $\mathcal{F}|_L$ (as defined in Lemma 2.3.2.8) is in ${}^{\overline{p}}\mathbf{Perv}(L)$, where $\overline{p}(T \cap L) = p(T)$ for any stratum*

T of X with $S \subset \overline{T}$. Moreover, if p is a GM-perversity, then $\overline{p}$ is also a GM-perversity up to an overall shift by $p(\dim(S)) + 1$.

Proof. Recall from the proof of Lemma 2.3.2.8 that $\mathcal{F}|_L \cong \pi_*(\mathcal{F}|_{V \smallsetminus S}) \cong \sigma^*(\mathcal{F}|_{V \smallsetminus S})$, where $V \cong \mathbb{R}^{\dim(S)} \times c(L)$ is a distinguished neighbourhood of some $x \in S$, the map $\pi : V \smallsetminus S \to L$ is the projection with fibre $\mathbb{R}^{\dim(S)} \times (0,1)$ and σ is a section of π. Note that σ is a normally nonsingular inclusion of codimension $\dim(S)+1$, see [GM83, Section 5.4]. For a stratum T of X such that $S \subset \overline{T}$, let $t : T \hookrightarrow X$ and $t_L : T \cap L \hookrightarrow L$ be the inclusions. Then

$$\begin{aligned} t_L^*(\mathcal{F}|_L) &\cong t_L^* \sigma^*(\mathcal{F}|_{V \smallsetminus S}) \\ &\cong \sigma^* t^*(\mathcal{F}|_{V \smallsetminus S}) \end{aligned}$$

and $t^*(\mathcal{F}|_{V \smallsetminus S}) \in \mathrm{D}^{\leq -p(T)}(T \cap V)$. Hence, we have $t_L^*(\mathcal{F}|_L) \in \mathrm{D}^{\leq -p(T)}(T \cap L)$. Similarly, using again that σ is normally nonsingular, we have

$$\begin{aligned} t_L^!(\mathcal{F}|_L) &\cong t_L^! \sigma^*(\mathcal{F}|_{V \smallsetminus S}) \\ &\cong t_L^! \sigma^!(\mathcal{F}|_{V \smallsetminus S})[\dim(S)+1] \\ &\cong \sigma^! t^!(\mathcal{F}|_{V \smallsetminus S})[\dim(S)+1] \\ &\cong \sigma^* t^!(\mathcal{F}|_{V \smallsetminus S}) \end{aligned}$$

and $t^!(\mathcal{F}|_{V \smallsetminus S}) \in \mathrm{D}^{\geq -p(T)}(T \cap V)$ so that $t_L^!(\mathcal{F}|_L) \in \mathrm{D}^{\geq -p(T)}(T \cap L)$. We conclude that $\mathcal{F}|_L \in {}^{\overline{p}}\mathbf{Perv}(L)$ where $\overline{p}(T \cap L) = p(T)$. If p is a GM-perversity and since $\dim(T \cap L) = \dim(T) - \dim(S) - 1$, we deduce that $\overline{p}$ is decreasing. Moreover, since

$$\begin{aligned} \overline{p}^*(T \cap L) &= -p(T \cap L) - \dim(T \cap L) \\ &= -p(T) - \dim(T) + \dim(S) + 1 \\ &= p^*(T) + \dim(S) + 1 \end{aligned}$$

we see that $\overline{p}^*$ is decreasing as well. Hence, up to an overall shift of $p(\dim(S)) + 1$, $\overline{p}$ is also a GM-perversity. □

Corollary 2.3.6.4. *Let p be a GM-pervrsity and $\mathcal{F} \in {}^{p}\mathbf{Perv}(X)$. Then*

$$H^k(L; \mathcal{F}|_L) = 0 \quad \text{if} \quad \begin{cases} k < \overline{p}(\dim(L)) = p(\dim(X)) \\ k > \overline{p}(\dim(L)) + \dim(L) = p(\dim(X)) + \operatorname{codim}(S) - 1 \end{cases},$$

where $\overline{p}(L \cap T) = p(T)$ for all strata T of X with $S \subset \overline{T}$.

2.3.7 Examples

In this section, we give some important examples and we relate perverse sheaves to other well-known mathematical structures.

Example 2.3.7.1 (Space with a single stratum)**.** *Let X be a topologically stratified space with a single stratum S, that is X is unstratified, see Example 2.3.1.5.i). Then the category of perverse sheaves on X is equivalent (up to a shift given by the value of the perversity on S) to the category of local systems, that is*

$$^p\mathbf{Perv}(X) \cong \mathbf{Loc}(X)[-p(S)].$$

Note that, if the hypotheses of Proposition 2.3.2.2 are satisfied, then we also have the equivalence $^p\mathbf{Perv}(X) \cong \mathbf{rep}(\pi_1(X))$.

Example 2.3.7.2 (Perverse Sheaves on a Point)**.** *If $X = \{\mathrm{pt}\}$, the conditions (2.4) imply the equivalence*

$$^p\mathbf{Perv}(X) \cong \mathbf{Vect}_{\mathbb{k}},$$

where $\mathbf{Vect}_{\mathbb{k}}$ denotes the category of finite dimensional $\mathbb{k}$-vector spaces over a field $\mathbb{k}$.

Example 2.3.7.3 (Perverse Sheaves for the Zero Perversity)**.** *Let us consider the zero perversity, see Example 2.3.3.4, on a topologically stratified space X. Then, the t-structure of (2.4) reduces to the standard t-structure on $\mathbf{D}_c(X)$, hence, see Definition 2.3.2.4, we get*

$$^o\mathbf{Perv}(X) \simeq \mathbf{Constr}(X).$$

In particular (2.5) implies that perverse functor for the zero perversity are functors of constructible sheaves. In this situation, the simple objects in $^o\mathbf{Perv}(X) \cong \mathbf{Constr}(X)$ have simpler description. Indeed, let $i_S : S \hookrightarrow X$ be the inclusion of a stratum into X and let $\mathcal{L} \in \mathbf{Loc}(S)$ be an irreducible local system. Then the simple perverse sheaves for the *zero perversity are*

$$\begin{aligned} {}^{o}\mathcal{S}_S &\cong {}^{0}i_{S!*}\mathcal{L} \\ &\cong \operatorname{im}({}^{0}i_{S!}\mathcal{L} \to {}^{0}i_{S*}) \quad \textit{by Definition } 2.3.5.4 \\ &\cong \operatorname{im}(i_{S!}\mathcal{L} \to i_{S*}\mathcal{L}) \\ &\cong i_{S!}\mathcal{L} \end{aligned}$$

as at the level of sheaves the map $i_{S!}\mathcal{L} \to i_{S*}\mathcal{L}$ is a monomorphism.

Example 2.3.7.4. *[Woo09, Section 3.1] Let us consider the projective complex line $X = \mathbb{P}^1$ stratified as in Example 2.3.1.6.iii). We denote the two strata by*

$$S_1 = U \cong \mathbb{C} \qquad \textit{and} \qquad S_0 = Z \cong \{\infty\}.$$

In this instance, there are three meaningful perversities $p(S) = (p(S_1), p(S_0))$ given by the top, middle and zero perversity, that is

$$t(S) = (-2, 0) \qquad m(S) = (-1, 0) \qquad o(S) = (0, 0).$$

Note that the first and last perversities are dual to each other, while the middle perversity is self-dual. The Auslander-Reiten quivers of ${}^{p}\mathbf{Perv}(X)$, see Remark 2.2.3.26, for the above perversities are respectively

$$\begin{tikzcd} & \Bbbk_X[2] \arrow[dr, two heads] & \\ i_*\Bbbk_Z \arrow[ur, hook] & & j_*\Bbbk_U[2] \arrow[ll, dashed] \end{tikzcd}$$

Figure 2.4: Auslander-Reiten quiver of ${}^{t}\mathbf{Perv}(\mathbb{P}^1)$

for the top perversity,

$$\begin{tikzcd}
& & \mathcal{M} \arrow[dr, two heads] & & \\
& j_!\Bbbk_U[1] \arrow[ur, hook] \arrow[dr, two heads] & & j_*\Bbbk_U[1] \arrow[ll, dashed] \arrow[dr, two heads] & \\
i_*\Bbbk_Z \arrow[ur, hook] & & \Bbbk_X[1] \arrow[ll, dashed] \arrow[ur, hook] & & i_*\Bbbk_Z \arrow[ll, dashed]
\end{tikzcd}$$

Figure 2.5: Auslander-Reiten quiver of ${}^m\mathbf{Perv}(\mathbb{P}^1)$

for the middle perversity, where $\mathcal{M}$ is the Beilinson's maximal extension discussed in Section 2.3.5, and

$$\begin{tikzcd}
& \Bbbk_X \arrow[dr, two heads] & \\
j_!\Bbbk_U \arrow[ur, hook] & & i_*\Bbbk_Z \arrow[ll, dashed]
\end{tikzcd}$$

Figure 2.6: Auslander-Reiten quiver of ${}^0\mathbf{Perv}(\mathbb{P}^1)$

for the zero perversity. Note also that the latter case correspond to the case of $\mathbf{Constr}(\mathbb{P}^1)$*, see Example 2.3.7.2, that is corresponds to constructible sheaves on* $\mathbb{P}^1$*. Dually, the first case describes the case of constructible co-sheaves on* $\mathbb{P}^1$*.*

Chapter 3

Projective Perverse Sheaves

In this chapter we present the results we achieved regarding projective covers of simple perverse sheaves and the characterisation of the category of p-perverse sheaves on a topologically stratified space X with finitely many strata, each with finite fundamental group, as a module category over a finite dimensional algebra and as representations of a quiver with relations.

In Section 3.1, we analyse the situation of projective local systems. The main reason is that the category ${}^p\mathbf{Perv}(X)$ is constructed by glueing local systems on strata with a shift given by the value of the perversity. Therefore, the case of local system serves both as the easiest example of perverse sheaves and as the situation we want to generalise.

In Section 3.2, we consider complementary open and closed unions of strata U and Z of a topologically stratified space, that is complementary maps $U \overset{j}{\hookrightarrow} X \overset{i}{\hookleftarrow} Z$, and we study which perverse functors preserve projective objects. We also note that, since the category ${}^p\mathbf{Perv}(X)$ of p-perverse sheaves is finite length, Noetherian and Artinian, Verdier duality, see Theorem 2.3.4.5, gives us dual statements regarding injective objects.

Section 3.3 is devoted to the construction of projective covers. Since we consider a topologically stratified space X with finitely many strata, each with finite fundamental group, the category ${}^p\mathbf{Perv}(X)$ has finitely many simple objects. Those can be either an intermediate extension of a simple arising from the open part U or an extension by zero of a simple object arising from the closed part Z. We therefore divide the construction in two parts, depending on the support of the considered simple object. Since the category ${}^p\mathbf{Perv}(X)$ is finite length, the construction of projective covers for simple objects implies that we can construct projective covers for any perverse sheaf. We also give examples of projective covers for the case of $X = \mathbb{P}^1$. We note that dualising the above construction gives us a procedure to build injective hulls for simple objects, and hence for any perverse sheaf. We then study which perverse functors preserve projective covers and dually injective hulls.

In Section 3.4, we present the results that we can deduce from the construction of projective covers and injective hulls. In Theorem 3.4.0.6, we prove equivalent characterisations of the category ${}^p\mathbf{Perv}(X)$ of p-perverse sheaves on a topologically stratified space X with finitely many strata, each with finite fundamental group. Note that Theorem 3.4.0.6 can be regarded as a generalisation of Proposition 3.1.0.3, which covers the case of local systems. We then underline important features of the presented results, for instance the independence from the perversity function and complex structure of the space.

In Section 3.5, we explain more consequences that follow from the results. For instance, the construction of projective covers and injective hulls allows us to build minimal projective (respectively injective) presentations and resolutions. This in turn gives a way to calculate the Auslander-Reiten translation in the Auslander-Reiten quiver of the category ${}^p\mathbf{Perv}(X)$.

Finally, in Section 3.6 we present some vanishing results for the Ext-groups in the constructible derived category $\mathbf{D}_\mathrm{c}(X)$ for a topologically stratified space X with finitely many strata, each with finite fundamental group. If one further assumes that ${}^p\mathbf{Perv}(X)$ is a faithful heart inside $\mathbf{D}_\mathrm{c}(X)$ and we restrict ourselves to GM-perversities, this implies a bound on the global dimension of perverse sheaves. In particular, we show that, under the above hypothesis, the global dimension is an integer between zero and the dimension of the space X.

3.1 Projective Local Systems

In this section, we first recall the definition of a locally connected and semi-locally simply connected space. We then characterise local systems on such spaces.

Definition 3.1.0.1. *Let X be a topological space and consider a point $x \in X$. We say that X is* **locally connected at** *x if for every open V containing x, there exists a connected open set U such that $x \in U \subset V$. The space X is locally connected if it is locally connected at x for every $x \in X$.*

Definition 3.1.0.2. *A topological space X is* **semi-locally simply connected** *if every* *point in X has a neighbourhood U such that every loop in U can be contracted to a point in X.*

Let S be a connected, locally connected, semi-locally simply connected space. Let us denote by $\mathbf{Loc}(S)$ the category of finite dimensional local system on S with coefficients in an algebraically closed field $\Bbbk$, that is, see Definition 2.3.2.1, the category of locally constant sheaves of finite dimensional $\Bbbk$-vector spaces on S.

Proposition 3.1.0.3. *The following are equivalent:*

i) $\pi_1(S)$ *is finite.*

ii) $\mathbf{Loc}(S) \simeq A$-**mod** *for a finite dimensional* $\Bbbk$*-algebra* A.

iii) $\mathbf{Loc}(S)$ *has a projective generator.*

Proof. Let $\pi_1(S)$ be finite, then $\Bbbk[\pi_1(S)]$ is a finite dimensional $\Bbbk$-algebra. The monodromy representation of a local system gives an exact equivalence $\mathbf{Loc}(S) \simeq \Bbbk[\pi_1(S)]$-**mod**. Conversely, suppose $\mathbf{Loc}(S) \simeq A$-**mod** for a finite dimensional $\Bbbk$-algebra A. Then $\Bbbk[\pi_1(S)]$ is Morita equivalent, see Definition 2.2.1.6, to A. Therefore the algebra A is also finite dimensional, implying that $\pi_1(S)$ is finite, see Remark 2.2.1.7.

Finally, since $\mathbf{Loc}(S)$ is a finite length $\Bbbk$-linear category (with finite dimensional morphism spaces), Proposition 2.2.1.5 gives the equivalence between the second and third statements. $\square$

Remark 3.1.0.4. *Since the category of perverse sheaves ${}^p\mathbf{Perv}(X)$ on a topologically stratified space X is the result of glueing local systems on its strata (with a shift prescribed by the value of the perversity on each stratum) see Remark 2.3.4.4, Proposition 3.1.0.3 can be regarded as the one we aim to generalise.*

The following observation explains the connection between the finiteness of the fundamental group of a space and the semi-simplicity of the category of perverse sheaves on it.

Remark 3.1.0.5. *Let S be a stratum of topologically stratified space X. There are equivalences*

$$\begin{aligned} {}^p\mathbf{Perv}(S) &\simeq \mathbf{Loc}(S)[-p(S)] \\ &\simeq \Bbbk[\pi_1(S)]\text{-}\mathbf{mod}, \end{aligned}$$

see Example 2.3.7.1. Therefore, if $\pi_1(S)$ is finite and the characteristic of the field $\Bbbk$ does not divide the order of $\pi_1(S)$, the category ${}^p\mathbf{Perv}(S)$ is semisimple by Maschke's Theorem.

3.2 Preservation Under Functors

In this section, we focus on the conditions under which a functor preserves projective objects in the category of p-perverse sheaves. In what follows, let U and Z denote complementary open and closed union of strata respectively, so that we have complementary maps

$$U \overset{j}{\hookrightarrow} X \overset{i}{\hookleftarrow} Z.$$

We begin with a well-known general result which holds for abelian categories.

Lemma 3.2.0.1. *Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories. Consider functors $F : \mathcal{A} \to \mathcal{B}$ and $G : \mathcal{B} \to \mathcal{A}$ such that:*

i) G is exact.

ii) F is left adjoint to G.

If $A \in \mathcal{A}$ is projective then $F(A) \in \mathcal{B}$ is projective.

Proof. By using the adjunction $F \dashv G$, we have

$$\mathrm{Hom}_{\mathcal{B}}(F(A), -) \cong \mathrm{Hom}_{\mathcal{A}}(A, G(-)) \cong \mathrm{Hom}_{\mathcal{A}}(A, -) \circ G.$$

G is exact by hypothesis and $\mathrm{Hom}_{\mathcal{A}}(A, -)$ is exact since $A \in \mathcal{A}$ is projective. Hence $\mathrm{Hom}_{\mathcal{B}}(F(A), -)$ is exact since composition of exact functors. This implies that $F(A) \in \mathcal{B}$ is projective. □

In Section 2.3.5 we saw that the six functor formalism on the constructible derived category $\mathbf{D}_{\mathrm{c}}(X)$ descends to perverse sheaves. As a consequence of the above lemma we then have the following result.

Lemma 3.2.0.2. *The functors ${}^p j_!$ and ${}^p i^*$ preserve projective perverse sheaves.*

Proof. The functors ${}^p j_!$ and ${}^p i^*$ are left adjoint to the exact functors j^* and i_* respectively, see (2.6). Therefore Lemma 3.2.0.1 implies the claim. □

In general, an exact functor does not preserve projective objects. In fact, if we start with a projective perverse sheaf on X and we restrict it to the open part, we need to impose an extra condition to get a projective perverse sheaf on U.

Lemma 3.2.0.3. *Let $\mathcal{P} \in {}^p\mathbf{Perv}(X)$ be a projective object such that ${}^pi^*\mathcal{P} = 0$. Then $j^*\mathcal{P} \in {}^p\mathbf{Perv}(U)$ is projective.*

Proof. In order to establish the claim it is enough to show that the functor

$$\mathrm{Hom}_{{}^p\mathbf{Perv}(U)}(j^*\mathcal{P}, -) \cong \mathrm{Hom}_{{}^p\mathbf{Perv}(X)}(\mathcal{P}, {}^pj_*(-))$$

is exact. Let us consider a short exact sequence of the form

$$0 \to \mathcal{E} \to \mathcal{F} \to \mathcal{G} \to 0$$

in ${}^p\mathbf{Perv}(U)$. Then, applying the left exact functor ${}^pj_*$, we have an exact sequence

$$0 \to {}^pj_*\mathcal{E} \to {}^pj_*\mathcal{F} \to {}^pj_*\mathcal{G} \to i_*\mathcal{C} \to 0$$

for some $\mathcal{C} \cong \mathrm{coker}({}^pj_*\mathcal{F} \to {}^pj_*\mathcal{G}) \in {}^p\mathbf{Perv}(X \smallsetminus U)$. Applying the exact functor $\mathrm{Hom}_{{}^p\mathbf{Perv}(X)}(\mathcal{P}, -)$ to the above exact sequence yields an exact sequence

$$0 \to \mathrm{Hom}(\mathcal{P}, {}^pj_*\mathcal{E}) \to \mathrm{Hom}(\mathcal{P}, {}^pj_*\mathcal{F}) \to \mathrm{Hom}(\mathcal{P}, {}^pj_*\mathcal{G}) \to \mathrm{Hom}(\mathcal{P}, i_*\mathcal{C}) \to 0.$$

The condition ${}^pi^*\mathcal{P} = 0$ implies that

$$\mathrm{Hom}_{{}^p\mathbf{Perv}(X)}(\mathcal{P}, i_*(-)) \cong \mathrm{Hom}_{{}^p\mathbf{Perv}(Z)}({}^pi^*\mathcal{P}, -) = 0,$$

hence the statement follows. $\square$

We now give two examples in our context where an exact functor does not preserve projective objects.

Example 3.2.0.4. *We show that the exact functor extension by zero i_* does not send a projective perverse sheaf $\mathcal{P} \in {}^p\mathbf{Perv}(Z)$ to a projective object in ${}^p\mathbf{Perv}(X)$.*

i) In the setting of Example 2.3.1.6.ii), let us consider the extension by zero $i_*\Bbbk_0$ of the constant sheaf on a point, that is the skyscraper sheaf at the origin, for the middle *perversity. In* ${}^{m}\mathbf{Perv}(\mathbb{C})$*, the epimorphism*

$$j_*\Bbbk_{\mathbb{C}^*}[1] \twoheadrightarrow i_*\Bbbk_0$$

does not split, thus $i_*\Bbbk_0$ *is not projective in* ${}^{m}\mathbf{Perv}(\mathbb{C})$*.*

ii) In the setting of Example 2.3.1.6.iii), let us consider the extension by zero $i_*\Bbbk_Z$ *of the projective perverse sheaf* $\Bbbk_Z \in {}^{m}\mathbf{Perv}(Z)$*. We have that* $i_*\Bbbk_Z$ *is not projective, since there is a non split short exact sequence*

$$0 \to \Bbbk_X[1] \to j_*\Bbbk_U[1] \to i_*\Bbbk_Z \to 0,$$

that is $\mathrm{Ext}^1_{{}^{m}\mathbf{Perv}(X)}(i_*\Bbbk_Z, -) \neq 0$*.*

Since the category ${}^{p}\mathbf{Perv}(X)$ is finite length, Artinian and Noetherian, see Section 2.3.6, dualising statements for projective objects in ${}^{p}\mathbf{Perv}(X)$ yields results for injective objects in ${}^{p^*}\mathbf{Perv}(X)$. Therefore, one can easily state which functors (and under which hypotheses) preserve injective perverse sheaves.

Remark 3.2.0.5. *Lemma 3.2.0.1 has a dual statement in terms of injective objects. In fact, a functor which is right adjoint to an exact functor between abelian categories preserves injective objects. Therefore, dually to the situation described in Lemma 3.2.0.2, we have that the functors* ${}^{p}j_*$ *and* ${}^{p}i^!$ *always preserve injective objects. The dual statement of Lemma 3.2.0.3 holds as well. That is, if* $\mathcal{I} \in {}^{p}\mathbf{Perv}(X)$ *is an injective object such that* ${}^{p}i^!\mathcal{I} = 0$*, then* $j^*\mathcal{I} \in {}^{p}\mathbf{Perv}(U)$ *is injective. Finally, the exact functor* i_* *does not preserve injective objects either. Indeed, in the case of Example 2.3.1.6.ii), one can consider the extension by zero* $i_*\Bbbk_Z$ *of the injective perverse sheaf* $\Bbbk_Z \in {}^{m}\mathbf{Perv}(Z)$*. The existence of the non split short exact sequence*

$$0 \to i_*\Bbbk_Z \to j_!\Bbbk_U[1] \to \Bbbk_X[1] \to 0$$

in ${}^{m}\mathbf{Perv}(X)$*, see Example 2.3.7.4, implies that* $\mathrm{Ext}^1_{{}^{m}\mathbf{Perv}(X)}(-, i_*\Bbbk_Z) \neq 0$*, that is* $i_*\Bbbk_Z$ *is not injective.*

3.3 Construction of Projective Covers

In this section we present the procedure which allows us to build projective covers for simple objects in ${}^{p}\mathbf{Perv}(X)$. Let X be a topologically stratified space with finitely many strata S, each with finite fundamental group. We will denote by U and Z complementary open and closed unions of strata respectively, that is there are complementary maps $j : U \hookrightarrow X$ and $i : Z \hookrightarrow X$. Under the considered hypothesis, the category ${}^p\mathbf{Perv}(X)$ has finitely many simple objects which are either an intermediate extension of a simple object arising from U or an extension by zero of a simple object arising from Z, see Section 2.3.6. Therefore, we divide the construction of projective covers in two cases depending on the form of the considered simple object. We will use induction on the number of strata, that is assuming we know the projective covers of simple objects in ${}^p\mathbf{Perv}(U)$ and ${}^p\mathbf{Perv}(Z)$, we want to characterise projective covers in ${}^p\mathbf{Perv}(X)$.

3.3.1 Projective Covers of Simple Objects Arising from the Open Part

Let us consider a stratum $T \subset U$ and a local system $\mathcal{L} \in \mathbf{Loc}(T)$. In this section, we characterise the projective cover of a simple object of the form ${}^p j_{!*}(\mathcal{S}_\mathcal{L}|_U)$, where we denote by $\mathcal{S}_\mathcal{L}|_U \in {}^p\mathbf{Perv}(U)$ the restriction of a simple object on X to a simple object on U. In particular, we show that the extension under the functor ${}^p j_!$ of the projective cover $\mathcal{P}_\mathcal{L}|_U$ of $\mathcal{S}_\mathcal{L}|_U$ in ${}^p\mathbf{Perv}(U)$, which we can assume constructed by inductive hypothesis on the number of strata, is the projective cover of the simple object $\mathcal{S}_\mathcal{L}$ in ${}^p\mathbf{Perv}(X)$.

Proposition 3.3.1.1. *Let $\mathcal{P}_\mathcal{L}|_U$ be the projective cover in ${}^p\mathbf{Perv}(U)$ of a simple object $\mathcal{S}_\mathcal{L}|_U \in {}^p\mathbf{Perv}(U)$. Then, ${}^p j_!(\mathcal{P}_\mathcal{L}|_U)$ is the projective cover of the simple object ${}^p j_{!*}(\mathcal{S}_\mathcal{L}|_U)$ in ${}^p\mathbf{Perv}(X)$.*

Proof. Since $\mathcal{P}_\mathcal{L}|_U$ is the projective cover of $\mathcal{S}_\mathcal{L}|_U$ in ${}^p\mathbf{Perv}(U)$ there is an epimorphism of the form $\mathcal{P}_\mathcal{L}|_U \twoheadrightarrow \mathcal{S}_\mathcal{L}|_U$ in ${}^p\mathbf{Perv}(U)$. Therefore, using that ${}^p j_!$ is right exact and Remark 2.3.5.5.ii) we have that the composite

$$ {}^p j_!(\mathcal{P}_\mathcal{L}|_U) \twoheadrightarrow {}^p j_!(\mathcal{S}_\mathcal{L}|_U) \twoheadrightarrow {}^p j_{!*}(\mathcal{S}_\mathcal{L}|_U) $$

is an epimorphism. Let us consider the diagram

$$\begin{tikzcd} {}^p j_!(\mathcal{P}_\mathcal{L}|_U) \arrow[rr, "\alpha"] \arrow[dr, two heads] & & {}^p j_!(\mathcal{P}_\mathcal{L}|_U) \arrow[dl, two heads] \\ & {}^p j_{!*}(\mathcal{S}_\mathcal{L}|_U) & \end{tikzcd} .$$

Since $\mathcal{P}_\mathcal{L}|_U$ is the projective cover of $\mathcal{S}_\mathcal{L}|_U$ in ${}^p\mathbf{Perv}(U)$, then $j^*\alpha$ is an isomorphism.

Therefore, α is an isomorphism as ${}^p j_!$ is fully faithful, see Section 2.3.5. Hence ${}^p j_!(\mathcal{P}_\mathcal{L}|_U)$ is the projective cover of $\mathcal{S}_\mathcal{L}$ in ${}^p\mathbf{Perv}(X)$. $\square$

Example 3.3.1.2. *Let us consider $X = \mathbb{P}^1$ stratified by a point $Z = \{\mathrm{pt}\}$ and its open complement $U = X \smallsetminus Z$ as in Example 2.3.1.6.iii). Depending on the perversity one considers, we have:*

i) For the top perversity the simple object supported on U is ${}^t\mathcal{S}_U \cong j_*\Bbbk_U$ and its projective cover is ${}^t\mathcal{P}_U \cong \Bbbk_X$.

ii) For the middle perversity the simple object supported on U is ${}^m\mathcal{S}_U \cong \Bbbk_X[1]$ and its projective cover is ${}^m\mathcal{P}_U \cong j_!\Bbbk_U[1]$.

iii) For the zero perversity the simple object supported on U is also projective, that is ${}^o\mathcal{S}_U \cong {}^o\mathcal{P}_U \cong j_!\Bbbk_U$.

Dualising Proposition 3.3.1.1 we have an explicit characterisation of the injective hull of a simple object supported on U.

Proposition 3.3.1.3. *Let $\mathcal{I}_\mathcal{L}|_U$ be the injective hull in ${}^p\mathbf{Perv}(U)$ of a simple object $\mathcal{S}_\mathcal{L}|_U \in {}^p\mathbf{Perv}(U)$. Then, ${}^p j_*(\mathcal{I}_\mathcal{L}|_U)$ is the injective hull of the simple object ${}^p j_{!*}(\mathcal{S}_\mathcal{L}|_U)$ in ${}^p\mathbf{Perv}(X)$.*

Proof. The proof is completely dual to the one of Proposition 3.3.1.1. $\square$

The following Example is dual to Example 3.3.1.2.

Example 3.3.1.4. *Let us consider $X = \mathbb{P}^1$ stratified by a point $Z = \{\mathrm{pt}\}$ and its open complement $U = X \smallsetminus Z$ as in Example 2.3.1.6.iii). Depending on the perversity one considers, we have:*

i) For the top perversity the simple object supported on U is ${}^t\mathcal{S}_U \cong j_*\Bbbk_U$ and it is also injective.

ii) For the middle perversity the simple object supported on U is ${}^m\mathcal{S}_U \cong \Bbbk_X[1]$ and its injective hull is ${}^m\mathcal{I}_U \cong j_*\Bbbk_U[1]$.

iii) For the zero perversity the simple projective object supported on U is ${}^o\mathcal{S}_U \cong j_!\Bbbk_U$ and its injective hull is ${}^o\mathcal{I}_U \cong \Bbbk_X[1]$.

3.3.2 Projective Covers of Simple Objects Arising from the Closed Part

Let $\mathcal{L} \in \mathbf{Loc}(S)$ be a local system for a stratum $S \subset Z$ and let $\widehat{\mathcal{P}}_{\mathcal{L}}$ denote the projective cover of $\mathcal{S}_{\mathcal{L}}|_Z$ in ${}^p\mathbf{Perv}(Z)$. Let I be the set of (isomorphism classes of) irreducible local system on the strata of X. We have that I is finite by assumption and that we can write $I = I_U + I_Z$ as a disjoint union of the sets of irreducible local systems on the complementary unions of strata of U and Z respectively. Let us consider the perverse sheaf

$$\widetilde{\mathcal{P}}_{\mathcal{L}} = \bigoplus_{\mathcal{M} \in I_U} \mathcal{P}_{\mathcal{M}} \otimes \operatorname{Ext}^1(i_*\widehat{\mathcal{P}}_{\mathcal{L}}, \mathcal{S}_{\mathcal{M}})^\vee \in {}^p\mathbf{Perv}(X). \tag{3.1}$$

Note that $\widetilde{\mathcal{P}}_{\mathcal{L}}$ is the sum of $\dim \operatorname{Ext}^1(i_*\widehat{\mathcal{P}}_{\mathcal{L}}, \mathcal{S}_{\mathcal{M}})$ copies of $\mathcal{P}_{\mathcal{M}} \cong {}^p j_!(\mathcal{P}_{\mathcal{M}}|_U)$ for $\mathcal{M} \in I_U$, where $\mathcal{P}_{\mathcal{M}}|_U$ denotes the projective cover of the simple object $\mathcal{S}_{\mathcal{M}}|_U$ in ${}^p\mathbf{Perv}(U)$. Therefore, $\widetilde{\mathcal{P}}_{\mathcal{L}}$ is projective since it is a sum of projective objects.

Let $\pi : \widetilde{\mathcal{P}}_{\mathcal{L}} \to \mathcal{Q}_{\mathcal{L}}$ be such that $\mathcal{Q}_{\mathcal{L}}$ has maximal length amongst quotients of $\widetilde{\mathcal{P}}_{\mathcal{L}}$ in ${}^p\mathbf{Perv}(X)$ for which there exists a morphism $\epsilon \in \operatorname{Ext}^1(i_*\widehat{\mathcal{P}}_{\mathcal{L}}, \mathcal{Q}_{\mathcal{L}})$ inducing isomorphisms

$$\operatorname{Hom}(\mathcal{Q}_{\mathcal{L}}, \mathcal{S}_{\mathcal{N}}) \cong \operatorname{Ext}^1(i_*\widehat{\mathcal{P}}_{\mathcal{L}}, \mathcal{S}_{\mathcal{N}}) : \phi \mapsto \phi \circ \epsilon \quad \forall \mathcal{N} \in I. \tag{3.2}$$

Remark 3.3.2.1. *The object $\mathcal{Q}_{\mathcal{L}} \in {}^p\mathbf{Perv}(X)$ exists since $\widetilde{\mathcal{P}}_{\mathcal{L}}$ has finite length (as does every object in ${}^p\mathbf{Perv}(X)$) and the quotient*

$$\widetilde{\mathcal{Q}}_{\mathcal{L}} = \bigoplus_{\mathcal{M} \in I_U} \mathcal{S}_{\mathcal{M}} \otimes \operatorname{Ext}^1(i_*\widehat{\mathcal{P}}_{\mathcal{L}}, \mathcal{S}_{\mathcal{M}})^\vee \in {}^p\mathbf{Perv}(X)$$

has the required properties. Note that in this case, a suitable morphism is given by the sum of units

$$\operatorname{Ext}^1(i_*\widehat{\mathcal{P}}_{\mathcal{L}}, \widetilde{\mathcal{Q}}_{\mathcal{L}}) \cong \bigoplus_{\mathcal{M} \in I_U} \operatorname{Ext}^1(i_*\widehat{\mathcal{P}}_{\mathcal{L}}, \mathcal{S}_{\mathcal{M}}) \otimes \operatorname{Ext}^1(i_*\widehat{\mathcal{P}}_{\mathcal{L}}, \mathcal{S}_{\mathcal{M}})^\vee.$$

The object $\widetilde{\mathcal{Q}}_{\mathcal{L}}$ above is the minimal length quotient with the required property (3.2).

Let $\mathcal{P}_{\mathcal{L}} \in {}^p\mathbf{Perv}(X)$ be the extension of $\mathcal{Q}_{\mathcal{L}}$ corresponding to $\epsilon \in \operatorname{Ext}^1(i_*\widehat{\mathcal{P}}_{\mathcal{L}}, \mathcal{Q}_{\mathcal{L}})$. In particular, in $\mathbf{D}_{\mathrm{c}}(X)$ there is a triangle

$$i_*\widehat{\mathcal{P}}_{\mathcal{L}}[-1] \xrightarrow{\epsilon} \mathcal{Q}_{\mathcal{L}} \to \mathcal{P}_{\mathcal{L}} \to i_*\widehat{\mathcal{P}}_{\mathcal{L}} \tag{3.3}$$

and therefore a short exact sequence

$$0 \to \mathcal{Q}_{\mathcal{L}} \to \mathcal{P}_{\mathcal{L}} \to i_* \widehat{\mathcal{P}}_{\mathcal{L}} \to 0$$

in ${}^p\mathbf{Perv}(X)$.

Lemma 3.3.2.2. *The perverse sheaf $\mathcal{P}_{\mathcal{L}} \in {}^p\mathbf{Perv}(X)$ is projective and indecomposable.*

Proof. The claim is equivalent to showing that $\mathrm{Ext}^1_{{}^p\mathbf{Perv}(X)}(\mathcal{P}_{\mathcal{L}}, \mathcal{S}_{\mathcal{N}}) = 0$ for any $\mathcal{N} \in I$ and that

$$\mathrm{Hom}_{{}^p\mathbf{Perv}(X)}(\mathcal{P}_{\mathcal{L}}, \mathcal{S}_{\mathcal{N}}) \cong \begin{cases} \Bbbk & \text{if } \mathcal{N} \cong \mathcal{L} \\ 0 & \text{otherwise} \end{cases}. \tag{3.4}$$

In fact, the former condition implies that $\mathcal{P}_{\mathcal{L}}$ is projective, while the latter that it is indecomposable. If we apply the functor $\mathrm{Hom}_{{}^p\mathbf{Perv}(X)}(-, \mathcal{S}_{\mathcal{N}})$ to (3.3), we get the long exact sequence

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}(i_*\widehat{\mathcal{P}}_{\mathcal{L}}, \mathcal{S}_{\mathcal{N}}) \arrow[r, "\alpha_1"] & \mathrm{Hom}(\mathcal{P}_{\mathcal{L}}, \mathcal{S}_{\mathcal{N}}) \arrow[r, "\beta_1"] & \mathrm{Hom}(\mathcal{Q}_{\mathcal{L}}, \mathcal{S}_{\mathcal{N}}) \arrow[dll, "\gamma_1"] \\
 & \mathrm{Ext}^1(i_*\widehat{\mathcal{P}}_{\mathcal{L}}, \mathcal{S}_{\mathcal{N}}) \arrow[r] & \mathrm{Ext}^1(\mathcal{P}_{\mathcal{L}}, \mathcal{S}_{\mathcal{N}}) \arrow[r] & \mathrm{Ext}^1(\mathcal{Q}_{\mathcal{L}}, \mathcal{S}_{\mathcal{N}}) \arrow[dll, "\gamma_2"] \\
 & \mathrm{Ext}^2(i_*\widehat{\mathcal{P}}_{\mathcal{L}}, \mathcal{S}_{\mathcal{N}}) \arrow[r] & \dots &
\end{tikzcd} \tag{3.5}$$

The property (3.2) implies that γ_1 is an isomorphism, hence $\beta_1 = 0$ and

$$\mathrm{Hom}(\mathcal{P}_{\mathcal{L}}, \mathcal{S}_{\mathcal{N}}) \cong \mathrm{Hom}(i_*\widehat{\mathcal{P}}_{\mathcal{L}}, \mathcal{S}_{\mathcal{N}}) \cong \mathrm{Hom}(\widehat{\mathcal{P}}_{\mathcal{L}}, {}^p i^! \mathcal{S}_{\mathcal{N}}),$$

which yields (3.4) as $\widehat{\mathcal{P}}_{\mathcal{L}}$ is the projective cover of $\mathcal{S}_{\mathcal{L}}$ in ${}^p\mathbf{Perv}(Z)$ and

$${}^p i^! \mathcal{S}_{\mathcal{N}} = \begin{cases} \mathcal{S}_{\mathcal{N}}|_Z & \text{if } \mathcal{N} \in I_Z \\ 0 & \text{if } \mathcal{N} \in I_U \end{cases}.$$

The last part of the long exact sequence (3.5) is then

$$0 \to \mathrm{Ext}^1(\mathcal{P}_{\mathcal{L}}, \mathcal{S}_{\mathcal{N}}) \to \mathrm{Ext}^1(\mathcal{Q}_{\mathcal{L}}, \mathcal{S}_{\mathcal{N}}) \xrightarrow{\gamma_2} \mathrm{Ext}^2(i_*\widehat{\mathcal{P}}_{\mathcal{L}}, \mathcal{S}_{\mathcal{N}}) \to \dots$$

In order to conclude that $\mathcal{P}_{\mathcal{L}}$ is projective in ${}^p\mathbf{Perv}(X)$, it is enough to show that γ_2 is injective. Suppose $\phi \circ \epsilon = 0$ for some $\phi \in \operatorname{Ext}^1(\mathcal{Q}_{\mathcal{L}}, \mathcal{S}_{\mathcal{N}})$. We show that $\phi = 0$ by proving that $\phi \neq 0$ leads to a contradiction. We have a commutative diagram of the form

$$\begin{tikzcd}
 & & \widetilde{\mathcal{P}}_{\mathcal{L}} \arrow[dl, dashed, "\pi'"'] \arrow[d, "\pi"] \arrow[dr, "0"] & \\
\mathcal{S}_{\mathcal{N}} \arrow[r] & \mathcal{Q}'_{\mathcal{L}} \arrow[r] & \mathcal{Q}_{\mathcal{L}} \arrow[r, "\phi"] & \mathcal{S}_{\mathcal{N}}[1] \\
 & & i_*\widehat{\mathcal{P}}_{\mathcal{L}}[-1] \arrow[ul, dashed, "\epsilon'"] \arrow[u, "\epsilon"'] \arrow[ur, "0"'] &
\end{tikzcd}$$

with middle row the triangle induced from ϕ. Note that π' exists since $\widetilde{\mathcal{P}}_{\mathcal{L}}$ is projective while ϵ' exists because $\phi \circ \epsilon = 0$. Applying the functor $\operatorname{Hom}_{\mathbf{D}_c(X)}(-, \mathcal{S}_{\mathcal{M}})$ to the above diagram gives that composition with ϵ' induces an isomorphism

$$\operatorname{Hom}(\mathcal{Q}'_{\mathcal{L}}, \mathcal{S}_{\mathcal{M}}) \cong \operatorname{Ext}^1(i_*\widehat{\mathcal{P}}_{\mathcal{L}}, \mathcal{S}_{\mathcal{M}})$$

for every $\mathcal{M} \in I$ when $\phi \neq 0$. For $\mathcal{M} \neq \mathcal{N}$, this follows from property (3.2) and $\operatorname{Hom}(\mathcal{S}_{\mathcal{N}}[1], \mathcal{S}_{\mathcal{M}}) \cong 0 \cong \operatorname{Hom}(\mathcal{S}_{\mathcal{N}}, \mathcal{S}_{\mathcal{M}})$. For $\mathcal{M} \cong \mathcal{N}$, it follows similarly but now using the fact that composing with ϕ induces an inclusion

$$\Bbbk \cong \operatorname{Hom}(\mathcal{S}_{\mathcal{N}}, \mathcal{S}_{\mathcal{N}}) \hookrightarrow \operatorname{Hom}(\mathcal{Q}_{\mathcal{L}}[-1], \mathcal{S}_{\mathcal{N}}) \cong \operatorname{Ext}^1(\mathcal{Q}_{\mathcal{L}}, \mathcal{S}_{\mathcal{N}}),$$

provided that $\phi \neq 0$. Therefore π' cannot be an epimorphism in ${}^p\mathbf{Perv}(X)$ if $\phi \neq 0$, for otherwise $\mathcal{Q}_{\mathcal{L}}$ would not be maximal in length amongst the quotients of $\widetilde{\mathcal{P}}_{\mathcal{L}}$ with the property (3.2), since $\mathcal{Q}'_{\mathcal{L}}$ has the property (3.2) and greater length. Therefore, since $\mathcal{S}_{\mathcal{N}}$ is simple we have

$$\operatorname{im}\pi' \cong \mathcal{Q}_{\mathcal{L}} \quad \text{and} \quad \mathcal{Q}'_{\mathcal{L}} \cong \mathcal{Q}_{\mathcal{L}} \oplus \mathcal{S}_{\mathcal{N}}.$$

Hence $\phi = 0$ after all and the map γ_2 is injective. □

Proposition 3.3.2.3. *The perverse sheaf $\mathcal{P}_{\mathcal{L}}$ is the projective cover of $\mathcal{S}_{\mathcal{L}}$ in ${}^p\mathbf{Perv}(X)$.*

Proof. We have that $\mathcal{P}_{\mathcal{L}} \in {}^p\mathbf{Perv}(X)$ is projective and indecomposable by Lemma 3.3.2.2, and there are epimorphisms

$$\mathcal{P}_{\mathcal{L}} \twoheadrightarrow i_*\widehat{\mathcal{P}}_{\mathcal{L}} \twoheadrightarrow \mathcal{S}_{\mathcal{L}}$$

in ${}^p\mathbf{Perv}(X)$. Then, the claim follows by Lemma 2.1.2.6. □

Example 3.3.2.4. *Let $X = \mathbb{P}^1$ stratified by a point $Z = \{\mathrm{pt}\}$ and its open complement $U \cong X \smallsetminus Z$ as in Example 2.3.1.6.iii) and consider the middle perversity. We want to construct the projective cover of $\mathcal{S}_Z \cong i_*\Bbbk_Z$ in ${}^m\mathbf{Perv}(X)$. First of all, we need to identify the object $\widetilde{\mathcal{P}}_Z$. Since we have*

$$i_*\widehat{\mathcal{P}}_Z \cong i_*\mathcal{S}_Z$$

it follows that $\mathrm{Ext}^1(i_*\widehat{\mathcal{P}}_Z, \mathcal{P}_U) \cong \Bbbk$. Therefore

$$\widetilde{\mathcal{P}}_Z \cong \mathcal{P}_U \otimes \mathrm{Ext}^1(\mathcal{S}_Z, \mathcal{S}_U) \cong \mathcal{P}_U.$$

Now, we need to find the biggest quotient (in length) $\mathcal{Q}_Z$ of $\widetilde{\mathcal{P}}_Z$ which satisfies (3.2). We claim that $\mathcal{Q}_Z \cong \widetilde{\mathcal{P}}_U$. By applying the functor $\mathrm{Ext}^i_{{}^m\mathbf{Perv}(X)}(\mathcal{S}_Z, -)$ to the exact sequence

$$0 \to \mathcal{S}_Z \to \mathcal{P}_U \to \mathcal{S}_U \to 0$$

we get the long exact sequence

$$\ldots \to \mathrm{Ext}^1(\mathcal{S}_Z, \mathcal{S}_Z) \to \mathrm{Ext}^1(\mathcal{S}_Z, \mathcal{P}_U) \to \mathrm{Ext}^1(\mathcal{S}_Z, \mathcal{S}_U) \to \mathrm{Ext}^2(\mathcal{S}_Z, \mathcal{S}_Z) \to \ldots$$

The first and last terms in the above long exact sequence are zero, as $\mathrm{Ext}^i(\mathcal{S}_Z, \mathcal{S}_Z) \cong 0$ for any $i \geq 1$, therefore

$$\mathrm{Ext}^1(\mathcal{S}_Z, \mathcal{P}_U) \cong \mathrm{Ext}^1(\mathcal{S}_Z, \mathcal{S}_U) \cong \Bbbk.$$

Let $0 \neq \epsilon \in \mathrm{Ext}^1(i_*\widehat{\mathcal{P}}_Z, \mathcal{Q}_Z) \cong \mathrm{Ext}^1(\mathcal{S}_Z, \mathcal{P}_U)$. Moreover

$$\mathrm{Hom}(\mathcal{Q}_Z, \mathcal{S}_\mathcal{N}) \cong \mathrm{Hom}(\mathcal{P}_U, \mathcal{S}_\mathcal{N}) \cong \begin{cases} \Bbbk & \text{if } \mathcal{N} \in \mathbf{Loc}(U) \\ 0 & \text{otherwise} \end{cases}$$

and

$$\mathrm{Ext}^1(i_*\widehat{\mathcal{P}}_Z, \mathcal{S}_\mathcal{N}) \cong \mathrm{Ext}^1(\mathcal{S}_Z, \mathcal{S}_U) \cong \begin{cases} \Bbbk & \text{if } \mathcal{N} \in \mathbf{Loc}(U) \\ 0 & \text{otherwise} \end{cases}$$

so the isomorphisms of (3.2) hold. The projective cover $\mathcal{P}_Z$ of $\mathcal{S}_Z \in {}^m\mathbf{Perv}(X)$ then sits in the short exact sequence

$$0 \to \mathcal{P}_U \to \mathcal{P}_Z \to \mathcal{S}_Z \to 0 \tag{3.6}$$

induced by the triangle (3.3). Therefore $\mathcal{P}_Z \cong \mathcal{M}$ in the notation of Example 2.3.7.4.

Remark 3.3.2.5. *Note that, since* ${}^p\mathbf{Perv}(X)$ *is a finite length category, see Section 2.3.6, the construction of projective covers for simple objects implies that every object* $\mathcal{E} \in {}^p\mathbf{Perv}(X)$ *has a projective cover* $\mathcal{P}(\mathcal{E}) \in {}^p\mathbf{Perv}(X)$*. In particular,* $\mathcal{P}(\mathcal{E})$ *is given by the sum of the projective covers of the semisimple object* $\mathrm{top}(\mathcal{E}) \cong \mathcal{E}/\mathrm{rad}(\mathcal{E})$*.*

Remark 3.3.2.6. *Note that the construction of projective covers for a simple object* $\mathcal{S}_{\mathcal{L}} \in {}^p\mathbf{Perv}(X)$ *arising from the closed part gets simpler if one consider the case of* Z *a single closed stratum. For instance, the projective perverse sheaf of (3.1) becomes*

$$\widetilde{\mathcal{P}}_{\mathcal{L}} = \bigoplus_{\mathcal{M} \in I_U} \mathcal{P}_{\mathcal{M}} \otimes \mathrm{Ext}^1(\mathcal{S}_{\mathcal{L}}, \mathcal{S}_{\mathcal{M}})^{\vee},$$

that is $\widetilde{\mathcal{P}}_{\mathcal{L}}$ *is the sum of all projective covers of simple objects arising from* U *for which there is an arrow* $\mathcal{L} \to \mathcal{M}$ *in the* Ext*-quiver of* ${}^p\mathbf{Perv}(X)$*, where* $\mathcal{L} \in I_Z$*.*

Remark 3.3.2.7. *Dualising the above construction gives a procedure which allows one to construct injective hulls for simple objects and hence any object in* ${}^p\mathbf{Perv}(X)$*. In particular, in the case of* $X = \mathbb{P}^1$ *stratified by a point* $Z = \{\mathrm{pt}\}$ *and its open complement* $U \cong X \smallsetminus Z$ *as in Example 2.3.1.6.iii) with the middle perversity, the dualised procedure for the construction of the injective hull of* $\mathcal{S}_Z$ *in* ${}^m\mathbf{Perv}(X)$ *gives that* $\mathcal{I}_Z \cong \mathcal{M}$*. That is Beilinson's maximal extension* $\mathcal{M} \in {}^m\mathbf{Perv}(X)$ *is an injective-projective m-perverse sheaf.*

3.3.3 Preserving Projective Covers

In this section, we study under which hypotheses functors preserve projective covers.

Lemma 3.3.3.1. *The functor* ${}^pj_!$ *preserves projective covers when they exist in* ${}^p\mathbf{Perv}(U)$*.*

Proof. It follows directly form the fact that ${}^pj_!$ is fully faithful, see Section 2.3.5. □

The exact functor j^* preserves projective covers $\mathcal{P} \in {}^p\mathbf{Perv}(X)$ only if $\mathcal{P}$ has no quotients supported on Z.

Lemma 3.3.3.2. *Suppose* $\mathcal{P} \twoheadrightarrow \mathcal{E}$ *is the projective cover of* $\mathcal{E}$ *in* ${}^p\mathbf{Perv}(X)$ *and that* ${}^pi^*\mathcal{P} \cong 0$*. Then,* $j^*\mathcal{P} \twoheadrightarrow j^*\mathcal{E}$ *is the projective cover of* $j^*\mathcal{E}$ *in* ${}^p\mathbf{Perv}(U)$*.*

Proof. Let us consider the projective cover $\mathcal{P} \twoheadrightarrow \mathcal{E}$ in ${}^p\mathbf{Perv}(X)$. By applying the exact functor j^* we have an epimorphism

$$j^*\mathcal{P} \to j^*\mathcal{E}$$

in ${}^p\mathbf{Perv}(U)$, with $j^*\mathcal{P}$ projective by Lemma 3.2.0.3. We need to show that given a commutative diagram

$$\begin{tikzcd} j^*\mathcal{P} \arrow[rr, "\alpha"] \arrow[dr, two heads] & & j^*\mathcal{P} \arrow[dl, two heads] \\ & j^*\mathcal{E} & \end{tikzcd}$$

then α is an isomorphism. We have the following diagram

$$\begin{tikzcd} {}^pj_!j^*\mathcal{P} \arrow[rr, "{}^pj_!\alpha"] \arrow[d, two heads] \arrow[dr, two heads] & & {}^pj_!j^*\mathcal{P} \arrow[dl, two heads] \arrow[d, two heads] \\ \mathcal{P} \arrow[rr, bend right] \arrow[dr, two heads] & {}^pj_!j^*\mathcal{E} \arrow[d, two heads] & \mathcal{P} \arrow[dl, two heads] \\ & \mathcal{E} & \end{tikzcd} \tag{3.7}$$

where ${}^pj_!j^*\mathcal{P} \to \mathcal{P}$ is an epimorphism since ${}^pi^*\mathcal{P} = 0$. Hence, as $\mathcal{P}$ is projective, there is a splitting $\mathcal{P} \to {}^pj_!j^*\mathcal{P}$. Taking $\beta : \mathcal{P} \to \mathcal{P}$ to be the composite $\mathcal{P} \to {}^pj_!j^*\mathcal{P} \xrightarrow{{}^pj_!\alpha} {}^pj_!j^*\mathcal{P} \to \mathcal{P}$, we have $\alpha = j^*\beta$. Moreover, as $\mathcal{P} \twoheadrightarrow \mathcal{E}$ is the projective cover in ${}^p\mathbf{Perv}(X)$ and we have that the bottom triangle in (3.7) commutes, β is an isomorphism. Hence α is an isomorphism. $\square$

In a similar way, the right exact functor ${}^pi^*$ preserves projective covers $\mathcal{P} \in {}^p\mathbf{Perv}(X)$ only if the restriction of $\mathcal{P}$ to the open part U is zero.

Lemma 3.3.3.3. *Suppose $\mathcal{P} \twoheadrightarrow \mathcal{E}$ is the projective cover of $\mathcal{E}$ in ${}^p\mathbf{Perv}(X)$ and that $j^*\mathcal{P} \cong 0$. Then, ${}^pi^*\mathcal{P} \twoheadrightarrow {}^pi^*\mathcal{E}$ is the projective cover of ${}^pi^*\mathcal{E}$ in ${}^p\mathbf{Perv}(Z)$.*

Proof. Let us consider the projective cover $\pi : \mathcal{P} \twoheadrightarrow \mathcal{E}$ in ${}^p\mathbf{Perv}(X)$. By applying the right exact functor ${}^pi^*$ we have an epimorphism

$${}^pi^*\mathcal{P} \twoheadrightarrow {}^pi^*\mathcal{E}$$

in ${}^p\mathbf{Perv}(Z)$, with ${}^pi^*\mathcal{P}$ projective by Lemma 3.2.0.2. We need to show that given any commutative diagram

$$\begin{tikzcd} {}^pi^*\mathcal{P} \arrow[rr, "\alpha"] \arrow[dr, two heads, "{}^pi^*\pi"'] & & {}^pi^*\mathcal{P} \arrow[dl, two heads, "{}^pi^*\pi"] \\ & {}^pi^*\mathcal{E} & \end{tikzcd} \tag{3.8}$$

then α is an isomorphism. We can construct the following commutative diagram.

$$\begin{tikzcd}
\mathcal{P} \arrow[rrrr, "\beta"] \arrow[dr, two heads] \arrow[dddrr, bend right, "\pi"'] & & & & \mathcal{P} \arrow[dl, two heads] \arrow[dddll, bend left, "\pi"] \\
& i_*{}^p i^* \mathcal{P} \arrow[rr, "i_*\alpha"] \arrow[dr, two heads, "i_*{}^p i^* \pi"'] & & i_*{}^p i^* \mathcal{P} \arrow[dl, two heads, "i_*{}^p i^* \pi"] & \\
& & i_*{}^p i^* \mathcal{E} \arrow[d, "\cong"] & & \\
& & \mathcal{E} & &
\end{tikzcd}.$$

Here we use the fact that $j^*\mathcal{E} \cong 0$ since j^* is an exact functor, which implies $\mathcal{E} \cong i_*{}^p i^* \mathcal{E}$, and the fact that $\mathcal{P}$ is projective to obtain the morphism β making the upper square commute. The left and right squares are natural squares for the unit $1 \to i_*{}^p i^*$ of the left adjunction and the inner triangle is obtain by applying the exact functor i_* to (3.8). Since the outer triangle commutes and $\pi : \mathcal{P} \twoheadrightarrow \mathcal{E}$ is the projective cover, we deduce that β is an isomorphism. Hence, $\alpha = {}^p i^* \beta$ is also an isomorphism. $\square$

Remark 3.3.3.4. *In the same vein as Remark 3.2.0.5, one can dualise the statements of Lemma 3.3.3.1, 3.3.3.2 and 3.3.3.3 in order to establish which functors preserve injective hulls. In particular we have that ${}^p j_*$ always preserves injective hulls, j^* preserves injective hulls in* $\ker {}^p i^!$ *and finally ${}^p i^!$ preserves injective hulls in* $\ker j^*$.

3.4 Main Results

In this section we present a very convenient description of the category ${}^p\mathbf{Perv}(X)$ of p-perverse sheaves on a topologically stratified space X in terms of module categories and representations of a quiver with relations. Such characterisation is based on the construction of projective covers of Section 3.3. Let $\Bbbk$ be an algebraically closed field with characteristic not dividing the order of the fundamental group $\pi_1(S)$ for any stratum $S \subset X$.

Theorem 3.4.0.1. *Let X be a topologically stratified space with finitely many strata, each with finite fundamental group. Then each simple object in ${}^p\mathbf{Perv}(X)$ has a projective cover.*

Proof. By hypothesis, there are finitely many simple objects, see Section 2.3.6.g). The simple objects can arise either from the open or from the closed part, see Section 2.3.6.c). Proposition 3.3.1.1 and Section 3.3.2 provide the construction of projective covers for each of the two cases. Remark 3.3.2.5 shows how to extend the construction of projective cover to any perverse sheaf. $\square$

Theorem 3.4.0.2. *Let X be a topologically stratified space with finitely many strata S all with finite fundamental group. Then ${}^p\mathbf{Perv}(X)$ has enough injectives and projectives.*

Proof. By Section 2.3.6.d) ${}^p\mathbf{Perv}(X)$ is a finite length category. Theorem 3.4.0.1 shows that any simple object has a projective cover. As noted in Proposition 3.3.1.3 and Remark 3.3.2.7, the whole construction is independent from choice of the perversity, therefore, every simple object in ${}^{p^*}\mathbf{Perv}(X)$ has a projective cover. By duality, see Theorem 2.3.4.5, every simple object in ${}^p\mathbf{Perv}(X)$ has an injective hull. Hence the claim follows. $\square$

Remark 3.4.0.3. *Let X be a topologically stratified space with finitely many strata, each with finite fundamental group. Let $\{\mathcal{S}_\mathcal{L}\}$ be the finite set of (isomorphism classes of) simple objects and denote by $\{\mathcal{P}_\mathcal{L}\}$ and $\{\mathcal{I}_\mathcal{L}\}$ the corresponding sets of (isomorphism classes of) projective covers and injective hulls for $\mathcal{L} \in I$. The objects*

$$\mathcal{P} = \bigoplus_{\mathcal{L}\in I} \mathcal{P}_\mathcal{L} \qquad \text{and} \qquad \mathcal{I} = \bigoplus_{\mathcal{L}\in I} \mathcal{I}_\mathcal{L}$$

are respectively a projective generator and an injective cogenerator for the category ${}^p\mathbf{Perv}(X)$.

Theorem 3.4.0.4. *Let X be a topologically stratified space. There is an equivalence of categories*

$${}^p\mathbf{Perv}(X) \simeq A_p\text{-}\mathbf{mod}$$

where A_p is a finite dimensional $\Bbbk$-algebra if and only if X has finitely many strata, each with finite fundamental group.

Proof. If there is an exact equivalence, then ${}^p\mathbf{Perv}(X)$ has a projective generator. This implies that there are finitely many isomorphism classes of simple perverse sheaves, therefore there are finitely many strata. In addition, each simple perverse sheaf $\mathcal{S}_\mathcal{L} \in {}^p\mathbf{Perv}(X)$ has a projective cover $\mathcal{P}_\mathcal{L} \twoheadrightarrow \mathcal{S}_\mathcal{L}$. Let $j : U \hookrightarrow X$ and $i : Z = X \smallsetminus U \hookrightarrow X$ be complementary open and closed inclusions of unions of strata respectively. Let I_U denote the set of (isomorphism classes of) irreducible local systems on strata in U, then

$$\mathcal{P}_U = \bigoplus_{\mathcal{L}\in I_U} \mathcal{P}_\mathcal{L}$$

is projective with ${}^{p}i^{*}\mathcal{P}_U = 0$. Indeed, if $\mathcal{L} \in \mathbf{Loc}(S)$ with $S \subset U$, then each projective cover $\mathcal{P}_{\mathcal{L}} \twoheadrightarrow \mathcal{S}_{\mathcal{L}}$ satisfies $\mathrm{Hom}(\mathcal{P}_{\mathcal{L}}, \mathcal{S}_{\mathcal{M}}) = 0$ for $\mathcal{M} \neq \mathcal{L}$. Therefore

$$\mathrm{Hom}(\mathcal{P}_{\mathcal{L}}, i_*\mathcal{C}) \cong \mathrm{Hom}({}^{p}i^{*}\mathcal{P}_{\mathcal{L}}, \mathcal{C}) = 0$$

for any $\mathcal{C} \in {}^{p}\mathbf{Perv}(X \smallsetminus U)$. By Lemma 3.2.0.3 we have that $j^*\mathcal{P}_U \in {}^{p}\mathbf{Perv}(U)$ is projective; in particular, it is a projective generator for ${}^{p}\mathbf{Perv}(U)$. Moreover, if $i_S : S \hookrightarrow X$ is the inclusion of a closed stratum (in U), then by Lemma 3.2.0.2 we have that ${}^{p}i_S{}^{*}j^{*}\mathcal{P}_U$ is a projective generator for $\mathbf{Loc}(S)$. Choosing a suitable U for each stratum $S \subset X$ implies that each $\mathbf{Loc}(S)$ has a projective generator. Therefore, by Proposition 3.1.0.3 each $\pi_1(S)$ is finite. Conversely, if there are finitely many strata, each with finite fundamental group, Theorem 3.4.0.1 gives that each simple object has a projective cover. The projective generator for ${}^{p}\mathbf{Perv}(X)$ is given by the direct sum of the projective covers of simple objects, see Remark 3.4.0.3. $\square$

We now show, using a standard argument in representation theory, see [ARS97, ASS06], that there is an equivalence between the category of perverse sheaves and finitely generated modules over the path algebra $\Bbbk\mathrm{Q}_p(X)/\mathrm{I}_p(X)$.

Proposition 3.4.0.5. *Let X be a topologically stratified space with finitely many strata S, all with finite fundamental group. There is an equivalence*

$${}^{p}\mathbf{Perv}(X) \simeq \Bbbk\mathrm{Q}_p(X)/\mathrm{I}_p(X)\text{-mod}$$

such that $\Bbbk\mathrm{Q}_p(X)/\mathrm{I}_p(X)$ is finite dimensional, where $\mathrm{Q}_p(X)$ is a finite quiver and $\mathrm{I}_p(X)$ is an admissible ideal.

Proof. The direct summands of the projective generator, see Remark 3.4.0.3, are pairwise non-isomorphic. Therefore the algebra $\mathrm{End}(\mathcal{P})$ is basic, see Definition 2.2.1.4. As $\Bbbk$ is assumed to be algebraically closed, $\mathrm{End}(\mathcal{P}) \cong \Bbbk\mathrm{Q}_p(X)/\mathrm{I}_p(X)$ for a finite quiver $\mathrm{Q}_p(X)$ and admissible ideal $\mathrm{I}_p(X) \subset \Bbbk\mathrm{Q}_p(X)$ of its path algebra. In particular, as prescribed by Definition 2.2.2.6, the quiver $\mathrm{Q}_p(X)$ has a vertex for each simple perverse sheaf, that is each irreducible local system on a stratum; it has $\dim \mathrm{Ext}^1(\mathcal{S}_{\mathcal{L}}, \mathcal{S}_{\mathcal{M}})$ arrows from the vertex which corresponds to the local system $\mathcal{L}$ to the one that corresponds to the local system $\mathcal{M}$. Moreover, choosing a basis for each $\mathrm{Ext}^1(\mathcal{S}_{\mathcal{L}}, \mathcal{S}_{\mathcal{M}})$ uniquely determines an algebra homomorphism $\Bbbk\mathrm{Q}_p(X) \twoheadrightarrow \mathrm{End}(\mathcal{P})$ and therefore an ideal $\mathrm{I}_p(X)$ such that $\mathrm{End}(\mathcal{P}) \cong$

$\Bbbk Q_p(X)/I_p(X)$. □

Using Proposition 2.2.1.5, we shall organise all the previous results in the following Theorem.

Theorem 3.4.0.6. *Let $\Bbbk$ be an algebraically closed field. The following conditions are equivalent:*

i) *X is a topologically stratified space with finitely many strata S, such that fundamental group $\pi_1(S)$ is finite for any stratum $S \subset X$ and the characteristic of $\Bbbk$ does not divide the order of the fundamental group $\pi_1(S)$ for any stratum $S \subset X$.*

ii) ${}^p\mathbf{Perv}(X)$ *has enough projectives.*

iii) ${}^p\mathbf{Perv}(X)$ *has enough injectives.*

iv) ${}^p\mathbf{Perv}(X)$ *is equivalent to A_p-**mod** for a finite dimensional algebra A_p.*

v) ${}^p\mathbf{Perv}(X)$ *is equivalent to* $\Bbbk Q_p(X)/I_p(X)$-mod *where $Q_p(X)$ is a finite quiver and $I_p(X)$ is an admissible ideal.*

3.4.1 Comments

In this section we point out some features and important aspects regarding the results presented in Section 3.4.

Remark 3.4.1.1. *The construction of projective covers presented in Section 3.3, hence Theorem 3.4.0.6, does not depend on the perversity. Indeed, it works for any perversities as in Definition 2.3.3.1. We do not assumed our perversities to belong to the more restrictive class of Goresky-MacPherson perversities, see [GM80] and Remark 2.3.3.7.*

Remark 3.4.1.2. *The results presented in Section 3.4 work for a topologically stratified space, see Definition 2.3.1.1, with finitely many strata, each with finite fundamental group. We do not require X to have a complex structure and we do not restrict ourselves to algebraic varieties. For example, the result mentioned in Theorem 3.4.0.2 appears in [BGS96, Theorem 3.3.1] (and in [CPS93]) for the case of an algebraic variety and middle perversity.*

Remark 3.4.1.3. *The results of Section 3.4 use only information about the strata of the topologically stratified space X. Information about links intervenes for instance when one wants to determine the algebra A_p which appears in Theorem 3.4.0.4.*

Remark 3.4.1.4. *The results of Section 3.4 extend the situation of Proposition 3.1.0.3 regarding local systems. Note that one could expect such generalisation as the category of* p*-perverse sheaves* ${}^p\mathbf{Perv}(X)$ *is roughly speaking built by glueing together local systems on strata of* X *with a shift given by the perversity* p.

Remark 3.4.1.5. *Note that the field* $\Bbbk$ *needs to be algebraically closed only in Proposition 3.4.0.5 in order to have the isomorphism* $\mathrm{End}(\mathcal{P}) \cong \Bbbk\mathrm{Q}_p(X)/\mathrm{I}_p(X)$. *In the rest, the field* $\Bbbk$ *does not play any role, therefore it can be assumed to be any field with characteristic not dividing* $\pi_1(S)$ *for any stratum* S *of* X.

3.5 Further Consequences

Let X be a topologically stratified space with finitely many strata S, each with finite fundamental group and $\Bbbk$ be an algebraically closed field with characteristic not dividing the order of the fundamental group $\pi_1(S)$ for any stratum $S \subset X$. In such situation, the category of perverse sheaves is a module category over a finite dimensional algebra. Therefore, the following well-known result holds.

Lemma 3.5.0.1. *Let* A *be a finite dimensional algebra and consider a chain of epimorphism of the form*

$$P \overset{\pi}{\twoheadrightarrow} M \overset{\sigma}{\twoheadrightarrow} N$$

in A-**mod***, where* P *is the projective cover of* N. *Then* P *is the projective cover of* M.

Proof. Given a commuting diagram

$$\begin{tikzcd} P \arrow[rr, "\alpha"] \arrow[dr, two heads, "\pi"'] & & P \arrow[dl, two heads, "\pi"] \\ & M & \end{tikzcd}$$

we can extend it to

$$\begin{tikzcd} P \arrow[rr, "\alpha"] \arrow[dr, two heads, "\pi"'] \arrow[ddr, bend right] & & P \arrow[dl, two heads, "\pi"] \arrow[ddl, bend left] \\ & M \arrow[d, "\sigma"] & \\ & N & \end{tikzcd} .$$

Since $P \twoheadrightarrow N$ is the projective cover we see that α is an isomorphism. Hence $P \twoheadrightarrow M$ is also the projective cover. $\square$

Moreover, we can consider the radical of a perverse sheaf $\mathcal{E} \in {}^p\mathbf{Perv}(X)$, see Definition 2.1.2.15. We have

$$\operatorname{rad}(\mathcal{E}) = \bigcap \{\mathcal{F} \hookrightarrow \mathcal{E} \mid \mathcal{E}/\mathcal{F} \text{ is simple}\}.$$

The above facts, together with Lemma 2.1.2.18, are useful to find the projective cover of a non-simple object.

Example 3.5.0.2. *Let us consider $X = \mathbb{P}^1$ stratified as in Example 2.3.1.6.iii) with the middle perversity. There is a chain of epimorphisms $\mathcal{M} \twoheadrightarrow j_*\Bbbk_U[1] \cong \mathcal{I}_U \twoheadrightarrow i_*\Bbbk_Z \cong \mathcal{S}_Z$ and $\mathcal{M}$ is the projective cover of $\mathcal{S}_Z$ in ${}^m\mathbf{Perv}(\mathbb{P}^1)$. Lemma 3.5.0.1 implies that*

$$\mathcal{P}(\mathcal{I}_U) \cong \mathcal{P}_Z \cong \mathcal{M}.$$

Let us check this using Lemma 2.1.2.18. First of all, we need to find the radical of $\mathcal{I}_U$, that is the intersection of its maximal sub-objects. Since $\mathcal{I}_U$ has only one sub-object, it is easy to check that $\operatorname{rad}(\mathcal{I}_U) \cong \mathcal{S}_U$*. Therefore, we have*

$$\begin{aligned}\mathcal{P}(\mathcal{I}_U) &\cong \mathcal{P}(\operatorname{top}(\mathcal{I}_U)) \\ &\cong \mathcal{P}(\mathcal{I}_U/\operatorname{rad}(\mathcal{I}_U)) \\ &\cong \mathcal{P}(\mathcal{S}_Z) \\ &\cong \mathcal{P}_Z.\end{aligned}$$

3.5.1 Minimal Projective Presentations and Resolutions

Theorem 3.4.0.1 guarantees that any perverse sheaf $\mathcal{E} \in {}^p\mathbf{Perv}(X)$ has a projective cover. Let us denote it by $\pi_\mathcal{E} : \mathcal{P}(\mathcal{E}) \twoheadrightarrow \mathcal{E}$. Iterating the construction of projective covers yields the minimal projective resolution, see Definition 2.1.2.7, and hence a minimal projective presentation for any $\mathcal{E} \in {}^p\mathbf{Perv}(X)$. In particular, one can consider the short exact sequence

$$0 \to \ker \pi_\mathcal{E} \to \mathcal{P}(\mathcal{E}) \to \mathcal{E} \to 0$$

in ${}^p\mathbf{Perv}(X)$. Then, taking the projective cover of $\ker \pi_{\mathcal{E}}$ gives the next term in the minimal projective resolution of $\mathcal{E}$, that is

$$\mathcal{P}^\bullet_{\mathcal{E}} = \ldots \to \mathcal{P}(\ker \pi_{\mathcal{E}}) \xrightarrow{\pi'_{\mathcal{E}}} \mathcal{P}(\mathcal{E}).$$

Note that $\ker \pi_{\mathcal{E}} \in {}^p\mathbf{Perv}(X)$, therefore, in the case such object is not simple, one can use Lemma 2.1.2.18 to find its projective cover. One can keep applying the same method to the map $\pi'_{\mathcal{E}}$ to construct the next term. While this construction can continue indefinitely in the case that $\mathcal{E}$ has infinite projective dimension, the final result of this procedure will be the complete minimal projective resolution, if $\mathcal{E}$ it has finite projective dimension.

Example 3.5.1.1. *Let $X = \mathbb{P}^1$ stratified as in Example 2.3.1.6.iii) with the middle perversity on it, that is $p = m$. We want to construct the minimal projective resolution of the simple object $\mathcal{S}_U \cong {}^m j_{!*}\Bbbk_U[1] \cong \Bbbk_X[1]$. The projective cover of $\mathcal{S}_U$ in ${}^m\mathbf{Perv}(X)$ is $\mathcal{P}_U \cong j_!\Bbbk_U[1]$, see Example 3.3.1.2.ii). The kernel of the map $\pi_U : \mathcal{P}_U \twoheadrightarrow \mathcal{S}_U$ is*

$$\ker \pi_U \cong \mathcal{S}_Z,$$

hence

$$\mathcal{P}(\ker \pi_U) \cong \mathcal{P}(\mathcal{S}_Z) \cong \mathcal{P}_Z.$$

Thus, the beginning of the minimal projective resolution of $\mathcal{S}_U$ is

$$\mathcal{P}^\bullet_U = \ldots \to \mathcal{P}_Z \xrightarrow{\pi'_U} \mathcal{P}_U.$$

The kernel of the map π'_U is $\mathcal{P}_U$, therefore the complete minimal projective resolution of $\mathcal{S}_U$ is

$$\mathcal{P}^\bullet_U = \mathcal{P}_U \to \mathcal{P}_Z \xrightarrow{\pi'_U} \mathcal{P}_U,$$

since the map $\mathcal{P}_U \to \mathcal{P}_Z$ is a monomorphism.

We now want to construct the minimal projective resolution of the simple object $\mathcal{S}_Z \cong i_*\Bbbk_Z$. The projective cover of $\mathcal{S}_Z$ in ${}^m\mathbf{Perv}(X)$ is $\mathcal{P}_Z \cong \mathcal{M}$, see Example 3.3.2.4. The kernel of the map $\pi_Z : \mathcal{P}_Z \twoheadrightarrow \mathcal{S}_Z$ is

$$\ker \pi_Z \cong \mathcal{P}_U.$$

Since that object is already projective, we do not need to take its projective cover. Moreover, *one can note that the map*

$$\ker \pi_Z \cong \mathcal{P}_U \to \mathcal{P}_Z$$

is a monomorphism. Therefore, the minimal projective resolution of $\mathcal{S}_Z$ in ${}^m\mathbf{Perv}(X)$ is given by

$$\mathcal{P}_Z^\bullet = \mathcal{P}_U \xrightarrow{\pi'_Z} \mathcal{P}_Z$$

as π'_Z is a monomorphism.

3.5.2 Minimal Projective Presentations

Recall that a minimal projective presentation of a perverse sheaf $\mathcal{E} \in {}^p\mathbf{Perv}(X)$ is given by the exact three term sequence

$$\mathcal{P}' \to \mathcal{P}(\mathcal{E}) \to \mathcal{E} \to 0$$

where $\mathcal{P}(\mathcal{E})$ is the projective cover of $\mathcal{E}$ and $\mathcal{P}' \cong \mathcal{P}(\ker \pi_{\mathcal{E}})$.

Proposition 3.5.2.1. *Let* A *be a finite dimensional $\Bbbk$-algebra. Let us denote by $\{\mathcal{S}_i\}$, $\{\mathcal{P}_i\}$ and $\mathcal{P}_i^\bullet = \ldots \to \mathcal{P}'_i \to \mathcal{P}_i$ the set of (isomorphism classes of) simple objects, of projective covers of simple object and the minimal projective resolution of a simple object $\mathcal{S}_i$ respectively. Then, there is an isomorphism*

$$\mathrm{Ext}^1(\mathcal{S}_i, \mathcal{S}_j) \cong \mathrm{Hom}(\mathcal{P}'_i, \mathcal{S}_j).$$

Proof. See [Ben98, Proof of Proposition 2.4.3] or [ASS06, Proof of Lemma 2.1.2]. □

Using the above result, we have that the minimal projective presentation of a simple perverse sheaf $\mathcal{S}_{\mathcal{L}} \in {}^p\mathbf{Perv}(X)$ is given by

$$\mathcal{P}'_{\mathcal{L}} \to \mathcal{P}_{\mathcal{L}} \to \mathcal{S}_{\mathcal{L}} \to 0,$$

where $\mathcal{P}_{\mathcal{L}}$ is constructed using the methods of Proposition 3.3.1.1 and Section 3.3.2 while

$$\mathcal{P}'_{\mathcal{L}} \cong \bigoplus_{\mathcal{M} \in I} \mathcal{P}_{\mathcal{M}} \otimes \mathrm{Ext}^1(\mathcal{S}_{\mathcal{L}}, \mathcal{S}_{\mathcal{M}})^\vee. \tag{3.9}$$

Remark 3.5.2.2. *Note that the object $\mathcal{P}'_{\mathcal{L}}$ in (3.9) is the sum of projective cover of simple objects $\mathcal{S}_{\mathcal{M}}$ for which there is an arrow $\mathcal{S}_{\mathcal{L}} \to \mathcal{S}_{\mathcal{M}}$ in the* Ext*-quiver* $\mathrm{Q}_p(X)$ *of* ${}^p\mathbf{Perv}(X)$ *(with the correct multiplicity). Therefore, the minimal projective presentations of simple objects can be read directly from the quiver* $\mathrm{Q}_p(X)$*.*

3.5.3 Minimal Injective Resolutions

Dualising the construction of the minimal projective resolution of an object $\mathcal{E} \in {}^p\mathbf{Perv}(X)$ gives the minimal injective resolution in ${}^{p^*}\mathbf{Perv}(X)$. Indeed, let

$$ {}^p\mathcal{P}^\bullet_{\mathcal{E}} = \ldots \to {}^p\mathcal{P}'(\mathcal{E}) \to {}^p\mathcal{P}(\mathcal{E}) $$

be the minimal projective resolution in ${}^p\mathbf{Perv}(X)$ of an object $\mathcal{E} \in {}^p\mathbf{Perv}(X)$. Using the fact that the duality $\mathcal{D} : {}^p\mathbf{Perv}(X) \to {}^{p^*}\mathbf{Perv}(X)$, see Theorem 2.3.4.5, is such that

$$ \begin{aligned} \mathcal{D}\mathcal{E} &\in {}^{p^*}\mathbf{Perv}(X), \\ \mathcal{D}{}^p\mathcal{P}(\mathcal{E}) &\cong {}^{p^*}\mathcal{I}(\mathcal{E}) \in {}^{p^*}\mathbf{Perv}(X), \\ \mathcal{D}{}^p\mathcal{P}'(\mathcal{E}) &\cong {}^{p^*}\mathcal{I}'(\mathcal{E}) \in {}^{p^*}\mathbf{Perv}(X) \end{aligned} $$

we have that

$$ {}^{p^*}\mathcal{I}^\bullet_{\mathcal{D}\mathcal{E}} = {}^{p^*}\mathcal{I}(\mathcal{E}) \to {}^{p^*}\mathcal{I}'(\mathcal{E}) \to \ldots $$

is the minimal injective resolution in ${}^{p^*}\mathbf{Perv}(X)$ of the object $\mathcal{D}\mathcal{E} \in {}^{p^*}\mathbf{Perv}(X)$.

Example 3.5.3.1. *Let $X = \mathbb{P}^1$ stratified as in Example 2.3.1.6.iii) with the middle perversity on it, that is $p = m$. The minimal projective resolutions for the two simple objects in ${}^m\mathbf{Perv}(X)$ are, see Example 3.5.1.1*

$$ \begin{aligned} \mathcal{P}^\bullet_U &\cong {}^m\mathcal{P}_U \to {}^m\mathcal{P}_Z \to {}^m\mathcal{P}_U \\ \mathcal{P}^\bullet_Z &\cong {}^m\mathcal{P}_U \to {}^m\mathcal{P}_Z. \end{aligned} $$

Since the middle perversity is self-dual, the minimal injective resolutions for the two simple objects, which are also self-dual, are

$$ \begin{aligned} \mathcal{I}^\bullet_U &\cong {}^m\mathcal{I}_U \to {}^m\mathcal{I}_Z \to {}^m\mathcal{I}_U \\ \mathcal{I}^\bullet_Z &\cong {}^m\mathcal{I}_Z \to {}^m\mathcal{I}_U. \end{aligned} $$

3.5.4 Auslander-Reiten Translation

The construction of minimal projective presentations for simple objects can be used to compute the Auslander-Reiten translation for simple non-projective objects, see Definition 2.2.3.19. The Auslander-Reiten translation, in view of Theorem 3.4.0.4, is a functor

$$\tau : {}^{p}\mathbf{Perv}(X) \xrightarrow{(-)^t} {}^{p^*}\mathbf{Perv}(X) \xrightarrow{\mathcal{D}} {}^{p}\mathbf{Perv}(X)$$

where $\mathcal{P}$ is the projective generator of the category ${}^{p}\mathbf{Perv}(X)$, see Remark 3.4.0.3, while the functors involved in the definition of τ are $(-)^t = \mathrm{Hom}_{{}^{p}\mathbf{Perv}(X)}(-, \mathcal{P})$ and $\mathcal{D} = \mathrm{Hom}_{{}^{p^*}\mathbf{Perv}(X)}(-, \Bbbk)$ respectively.

Example 3.5.4.1. *Let us consider $X = \mathbb{P}^1$ stratified as in Example 2.3.1.6.iii) with the middle perversity (which is self-dual). From Example 2.3.7.4 we know that ${}^{m}\mathbf{Perv}(X)$ has five indecomposable objects. Amongst them three are non-projective, that is the two simple objects $\mathcal{S}_U$ and $\mathcal{S}_Z$ and the injective hull of the simple object supported on the open, namely $\mathcal{I}_U$. We want to calculate the Auslander-Reiten translation for these objects. First of all, we need projective presentations for them. Example 3.5.3.1 and Example 3.5.0.2 show that they are*

$$\mathcal{P}_Z \to \mathcal{P}_U \twoheadrightarrow \mathcal{S}_U$$
$$\mathcal{P}_U \hookrightarrow \mathcal{P}_Z \twoheadrightarrow \mathcal{S}_Z$$
$$\mathcal{P}_Z \to \mathcal{P}_Z \twoheadrightarrow \mathcal{I}_U.$$

Since we have

$$(\mathcal{P}_U)^t \cong (\mathcal{I}_U)^t \cong (\mathcal{S}_Z)^t \cong \mathcal{P}_U$$
$$(\mathcal{P}_Z)^t \cong \mathcal{P}_Z$$
$$(\mathcal{S}_U)^t \cong 0$$

then

$$\mathcal{D}(\mathcal{P}_U)^t \cong \mathcal{D}(\mathcal{I}_U)^t \cong \mathcal{D}(\mathcal{S}_Z)^t \mathcal{I}_U$$
$$\mathcal{D}(\mathcal{P}_Z)^t \cong \mathcal{P}_Z$$
$$\mathcal{D}(\mathcal{S}_U)^t \cong 0.$$

The exact sequences

$$0 \to \tau\mathcal{S}_U \to \mathcal{D}(\mathcal{P}_Z)^t \to \mathcal{D}(\mathcal{P}_U)^t \to \mathcal{D}(\mathcal{S}_U)^t \to 0$$
$$0 \to \tau\mathcal{S}_Z \to \mathcal{D}(\mathcal{P}_U)^t \to \mathcal{D}(\mathcal{P}_Z)^t \to \mathcal{D}(\mathcal{S}_Z)^t \to 0$$
$$0 \to \tau\mathcal{I}_U \to \mathcal{D}(\mathcal{P}_Z)^t \to \mathcal{D}(\mathcal{P}_Z)^t \to \mathcal{D}(\mathcal{I}_U)^t \to 0$$

then give

$$\tau\mathcal{S}_U \cong \mathcal{S}_Z, \qquad \tau\mathcal{S}_Z \cong \mathcal{S}_U \qquad \textit{and} \qquad \tau\mathcal{I}_U \cong \mathcal{P}_U.$$

Note that this agrees with Example 2.3.7.4.

3.6 On Global Dimension of Perverse Sheaves

In this section, let X be a topologically stratified space with finitely many strata, each with finite fundamental group and $\Bbbk$ an algebraically closed field with characteristic not dividing the order of the fundamental groups of strata. Let us denote by $i : S \hookrightarrow X$ the inclusion of a closed stratum with complementary map $j : U \hookrightarrow X$. Moreover, let us consider a geometric perversity p, see Remark 2.3.3.6, such that p and its dual p^* are both decreasing functions which depend only on the dimension. By Lemma 2.3.3.8, p and p^* are GM perversities.

Proposition 3.6.0.1. *Let X be a compact topologically stratified space, p a GM-perversity and $\mathcal{E} \in {}^p\mathbf{Perv}(X)$. Then*

$$H^k(X;\mathcal{E}) = 0 \quad \textit{for} \quad \begin{cases} k < p(\dim(X)) \\ k > p(\dim(X)) + \dim(X) = -p^*(\dim(X)) \end{cases}.$$

Proof. Recall that p is a GM-perversity when

$$-\dim(S) \leq p(S) \leq 0$$

for each stratum S of X and both p and its dual p^* are decreasing functions depending only on the dimension. In particular

$$p^*(\dim(X)) \leq p^*(S) \quad \text{and} \quad p(\dim(S)) \leq p(S)$$

for all strata S. It follows that

$$\mathrm{D}^{<p^*(\dim(X))}(S) \subset \mathrm{D}^{<p^*(S)} \quad \text{and} \quad \mathrm{D}^{<p(\dim(X))}(S) \subset \mathrm{D}^{<p(S)}(S)$$

for all strata S, and hence that

$${}^0\mathrm{D}^{<p^*(\dim(X))}(X) \subset {}^{p^*}\mathrm{D}^{<0} \quad \text{and} \quad {}^0\mathrm{D}^{<p(\dim(X))}(X) \subset {}^p\mathrm{D}^{<0}(X).$$

In particular, if $\pi : X \to \{\mathrm{pt}\}$ for any $\mathcal{F} \in \mathrm{D}^{<p^*(\dim(X))}(\{\mathrm{pt}\})$ we have $\pi^*\mathcal{F} \in^{p^*} \mathrm{D}^{<0}(X)$ and so by duality, for any $\mathcal{F} \in \mathrm{D}^{>-p^*(\dim(X))}(\{\mathrm{pt}\}) = \mathrm{D}^{>p(\dim(X))+\dim(X)}(\{\mathrm{pt}\})$ we have that $\pi^!\mathcal{F} \in^p \mathrm{D}^{>0}(X)$. Similarly, for any $\mathcal{G} \in \mathrm{D}^{<p(\dim(X))}(\{\mathrm{pt}\})$ we find $\pi^*\mathcal{G} \in^p \mathrm{D}^{<0}(X)$. Recalling that for a compact X the functor π_* is both left adjoint to $\pi^!$ and right adjoint to π^*, we note that

$$\mathrm{Hom}(\pi_*\mathcal{E}, \mathcal{F}) \cong \mathrm{Hom}(\mathcal{E}, \pi^!\mathcal{F}) = 0 \quad \forall \mathcal{F} \in \mathrm{D}^{>p(\dim(X))+\dim(X)}(\{\mathrm{pt}\})$$

and

$$\mathrm{Hom}(\mathcal{G}, \pi_*\mathcal{E}) \cong \mathrm{Hom}(\pi^*\mathcal{G}, \mathcal{E}) = 0 \quad \forall \mathcal{G} \in \mathrm{D}^{<p(\dim(X))}(\{\mathrm{pt}\}).$$

The claimed vanishing for $H^k(X;\mathcal{E}) = H^k(\pi_*\mathcal{E})$ follows immediately. $\square$

Remark 3.6.0.2. *Proposition 3.6.0.1 agrees with [Dim04, Proposition 5.2.12 and Corollary 5.2.14] for the (relative) case of analytic spaces.*

Remark 3.6.0.3. *Proposition 3.6.0.1 holds for any perversity p such that p and p^* are decreasing functions of the dimension (that is, we do not need to normalise so that p is a geometric perversity and hence GM).*

Remark 3.6.0.4. *Proposition 3.6.0.1 generalises the classical result for which $H^k(X;\mathcal{L}) = 0$ for $k < 0$ or $k > \dim(X)$ where $\mathcal{L} \in \mathbf{Loc}(X)$.*

Proposition 3.6.0.5. *Let p be a GM-perversity and $\mathcal{E}, \mathcal{F} \in {}^p\mathbf{Perv}(X)$. Then*

$$\mathrm{Ext}^k_{\mathbf{D}_c(X)}(\mathcal{E}, \mathcal{F}) = 0 \quad \text{if} \quad \begin{cases} k < 0 \\ k > \dim(X) \end{cases}.$$

Proof. When X has a single stratum, $\mathcal{E}$ and $\mathcal{F}$ are shifted local systems and $\mathrm{Ext}^k_{\mathbf{D}_c(X)}(\mathcal{E}, \mathcal{F}) \cong \mathrm{Ext}^k_{\mathbf{D}_c(X)}(\Bbbk_X, \mathcal{E}^\vee \otimes \mathcal{F}) = 0$ for $k < 0$ or $k > \dim(X)$ by Remark 3.6.0.4. Let us consider the triangle

$$i_*i^!\mathcal{F} \to \mathcal{F} \to j_*j^*\mathcal{F} \to i_*i^!\mathcal{F}[1]$$

and apply the functor $\mathrm{Ext}^k_{\mathbf{D}_c(X)}(\mathcal{E}, -)$ to get the long exact sequence

$$\ldots \to \mathrm{Ext}^k_{\mathbf{D}_c(X)}(\mathcal{E}, i_*i^!\mathcal{F}) \to \mathrm{Ext}^k_{\mathbf{D}_c(X)}(\mathcal{E}, \mathcal{F}) \to \mathrm{Ext}^k_{\mathbf{D}_c(X)}(\mathcal{E}, j_*j^*\mathcal{F}) \to \ldots \tag{3.10}$$

By adjunction, the right hand side in (3.10) becomes $\mathrm{Ext}^k_{\mathbf{D}_c(U)}(j^*\mathcal{E}, j^*\mathcal{F})$ for which we can use induction on the number of strata. Therefore, we need to understand the range for which the left hand side $\mathrm{Ext}^k_{\mathbf{D}_c(X)}(\mathcal{E}, i_*i^!\mathcal{F})$ in (3.10) vanishes. Let $\mathcal{F} \in {}^p\mathbf{Perv}(X)$ and consider the triangle

$$i^*i_*i^!\mathcal{F} \to i^*\mathcal{F} \to i^*j_*j^*\mathcal{F} \to i^*i_*i^!\mathcal{F}[1]$$

in $\mathbf{D}_c(S)$. Applying cohomology yields the long exact sequence

$$\ldots H^{k-1}(i^*\mathcal{F}) \to H^{k-1}(i^*j_*j^*\mathcal{F}) \to H^k(i^!\mathcal{F}) \to H^k(i^*\mathcal{F}) \to \ldots$$

which for $k > 0$ implies $H^k(i^!\mathcal{F}) \cong H^{k-1}(i^*j_*j^*\mathcal{F})$ since $i^*\mathcal{F} \in \mathrm{D}^{\leq -p(S)}(S)$ and $i^!\mathcal{F} \in \mathrm{D}^{\geq -p(S)}(S)$. Moreover, by Lemma 2.3.2.8, we have $H^k(i^!\mathcal{F}) \cong H^{k-1}(i^*j_*j^*\mathcal{F}) \cong H^{k-1}(L_S; \mathcal{F}|_{L_S})$ where L_S is the link of S. By Lemma 2.3.6.3, $\mathcal{F}|_{L_S}$ is perverse for a (shifted) GM-perversity. Hence, the above groups vanish outside a range of length $\dim(L_S) = \mathrm{codim}(S) - 1$. Combined with the fact that $i^*\mathcal{F} \in \mathrm{D}^{\leq -p(S)}(S)$ and $i^!\mathcal{F} \in \mathrm{D}^{\geq -p(S)}(S)$ we deduce by induction that

$$\mathrm{Ext}^l_{\mathbf{D}_c(X)}(\mathcal{E}, i_*i^!\mathcal{F}) \cong \mathrm{Ext}^l_{\mathbf{D}_c(S)}(i^*\mathcal{E}, i^!\mathcal{F}) = 0$$

for $l < 0$ or $l > \dim(S) + \mathrm{codim}(S) = \dim(X)$. Thus, the claim follows. $\square$

Definition 3.6.0.6. *Let $\mathcal{A}$ be an abelian category. The* **global dimension of** $\mathcal{A}$ *is either infinite or the supremum of the lengths of extensions between objects in $\mathcal{A}$, that is*

$$\mathrm{gldim}(\mathcal{A}) = \sup_d \{\mathrm{Ext}^d_{\mathcal{A}}(A, B) \quad \textit{for} \quad A, B \in \mathcal{A}\}.$$

Proposition 3.6.0.7. *Let ${}^p\mathbf{Perv}(X)$ be a faithful heart of $\mathbf{D}_c(X)$, then*

$$0 \leq \mathrm{gldim}({}^p\mathbf{Perv}(X)) \leq \dim(X).$$

Proof. The faithfulness of ${}^p\mathbf{Perv}(X)$ inside $\mathbf{D}_c(X)$, see Theorem 2.1.5.2 and Remark

2.1.5.3, implies that for any $k \in \mathbb{Z}$

$$\mathrm{Ext}^k_{\mathbf{D}_c(X)}(\mathcal{E}, \mathcal{F}) \cong \mathrm{Ext}^k_{{}^p\mathbf{Perv}(X)}(\mathcal{E}, \mathcal{F}).$$

Therefore, Proposition 3.6.0.5 implies the result. $\square$

Remark 3.6.0.8. *Proposition 3.6.0.7 remains true if one replaces the hypothesis of the faithfulness of* ${}^p\mathbf{Perv}(X)$ *with the slightly milder assumption*

$$\mathrm{Ext}^k_{{}^p\mathbf{Perv}(X)}(\mathcal{E}, \mathcal{F}) \hookrightarrow \mathrm{Ext}^k_{\mathbf{D}_c(X)}(\mathcal{E}, \mathcal{F})$$

for $k \leq \dim(X) - 1$.

Remark 3.6.0.9. *Let* X *be a topologically stratified space such that* $\dim(X) \leq 2$. *Then* ${}^p\mathbf{Perv}(X)$ *has finite global dimension.*

Example 3.6.0.10. *Let* $X = \mathbb{P}^1$ *stratified as in Example 2.3.1.6.iii) with the middle perversity on it, that is* $p = m$. *The heart* ${}^m\mathbf{Perv}(\mathbb{P}^1)$ *is faithful in* $\mathbf{D}_c(X)$ *by [Bei87b] so by above*

$$\mathrm{gldim}({}^m\mathbf{Perv}(\mathbb{P}^1)) \leq \dim_{\mathbb{R}}(\mathbb{P}^1) = 2,$$

and in fact $\mathrm{gldim}({}^m\mathbf{Perv}(\mathbb{P}^1)) = 2$ *in this case, see Example 3.5.1.1.*

The next observation shows that if the heart ${}^p\mathbf{Perv}(X)$ is faithful, then there exists a Serre functor.

Remark 3.6.0.11. *Let* X *be a topologically stratified space with finitely many strata, each with finite fundamental group, and* $\Bbbk$ *an algebraically closed field with characteristic that does not divide the order of the fundamental groups of strata. Theorem 3.4.0.6 says that there is an equivalence of categories*

$${}^p\mathbf{Perv}(X) \simeq A_p\text{-}\mathbf{mod},$$

where A_p *is a finite dimensional algebra. When the heart* ${}^p\mathbf{Perv}(X)$ *is faithful,* A_p *has also finite global dimension. Then, by [MS] there is a Serre functor on* $\mathbf{D}_c(X)$. *There exists a Serre functor, again by [MS], also in the situation of Remark 3.6.0.9.*

Chapter 4

Quiver Description of Perverse Sheaves

In this chapter, we study perverse sheaves as quiver representations. For a topologically stratified space X with finitely many strata S, each with finite fundamental group, and for an algebraically closed field $\Bbbk$ with characteristic not dividing the order of $\pi_1(S)$ for any stratum $S \subset X$, Proposition 3.4.0.5 and Theorem 2.2.2.14 give a characterisation of p-perverse sheaves on X as finitely generated (right) modules over the path algebra and as representations of a quiver with relations respectively. The quiver in question is the Ext-quiver defined in 2.2.2.6. This point of view is very interesting and fruitful as it allows us to translate questions about p-perverse sheaves into the language of linear algebra.

In Section 4.1 we explain how we can build the Ext-quiver $\mathrm{Q}_p(X)$ using only topological data of the space X. Indeed, the vertices are given by (isomorphism classes of) irreducible local systems on strata, that is by simple p-perverse sheaves. Arrows between simple objects are determined by the dimension of the Ext^1-groups between the corresponding simple objects. Moreover, we analyse how Ext-quivers relative to pairs of dual perversities behave under Verdier duality. Finally, we explain some examples. We study $X = \mathbb{P}^1$ stratified by a point and the open complement, see Example 2.3.1.6.iii), for the zero, middle and top perversity, the first quadrant stratified by the origin and one axis, see Example 2.3.1.6.i), for the zero perversity (which corresponds to constructible sheaves) and the circle stratified by a point and its open complement, see Example 4.1.2.6, for the zero perversity.

In Section 4.2, we deal with the ideal of relations $\mathrm{I}_p(X)$ of $\mathrm{Q}_p(X)$. In general, it is very hard to determine the whole ideal of relations because such information is encoded in an A_∞-structure rather than in the structure of the constructible derived category $\mathbf{D}_c(X)$. Although we cannot determine the ideal $I_p(X)$, we can determine its quadratic part, as such information is encoded in the structure of the ambient constructible derived category. In Section 4.2.3 we explain how to determine the quadratic part of relations inductively by adding one closed stratum at a time. Note that, for instance, this applies to quadratic algebras, a class of algebras that contains Koszul algebras and includes the case of the zero perversity, which corresponds to constructible sheaves.

In Section 4.3, we give a characterisation of higher Ext-groups between simple objects arising from the open part and the simple arising from the closed stratum we are adding (and the other way around) in terms of intersection cohomology groups of links. Note that this information, as explained in Section 4.2.3, is what is needed in order to be able to determine the quadratic part of relations.

In Section 4.4, we explain how to determine the quiver representation for simple perverse sheaves and for indecomposable projective objects, that is for projective covers of simple objects. We show how to do that in the case of $\mathbb{P}^1$ stratified by a point and the open complement with the middle perversity and for the case of the first quadrant stratified by the origin and one axis. In order to use this approach, one needs to know the ideal of relations $I_p(X)$ first.

In Section 4.5, we recall the general theory about the projective quiver $P_p(X)$ which is isomorphic to the Ext-quiver $Q_p(X)$. The Proj-quiver $P_p(X)$ can be constructed by using irreducible projective objects and (some) maps between them. Although this is very general, it turns out to be quite difficult in practice as it requires one to be able to control all the maps between irreducible projective objects.

4.1 Ext-quiver

Let X be a topologically stratified space with finitely many strata S, each with finite fundamental group and $\Bbbk$ an algebraically closed field such that $\mathrm{char}(\Bbbk)$ does not divide the order of $\pi_1(S)$ for any stratum $S \subset X$. Let p be a perversity on X and denote by ${}^p\mathbf{Perv}(X)$ the category of p-perverse sheaves on X. Proposition 3.4.0.5 gives the equivalence

$$ {}^p\mathbf{Perv}(X) \simeq \Bbbk Q_p(X)/I_p(X)\text{-mod}, \tag{4.1} $$

where $\mathrm{Q}_p(X)$ is the Ext-quiver, see Definition 2.2.2.6 and [Ben98, Definition 4.1.6] as well, and $\mathrm{I}_p(X)$ is the ideal of relations. The equivalence (4.1) can be equivalently written, see Theorem 2.2.2.14, as

$$ {}^p\mathbf{Perv}(X) \simeq \mathbf{rep}(\mathrm{Q}_p(X), \mathrm{I}_p(X)). \tag{4.2}$$

Our goal is to construct the quiver $\mathrm{Q}_p(X)$ and determine, at least in some important cases like that of quadratic algebras, the ideal of relations $\mathrm{I}_p(X)$.

4.1.1 Vertices and arrows of $\mathrm{Q}_p(X)$

In this section we characterise vertices and arrows of the quiver $\mathrm{Q}_p(X)$ of the category ${}^p\mathbf{Perv}(X)$. This can be achieved in purely topological terms, that is we can completely determine the quiver $\mathrm{Q}_p(X)$ using only topological data of the topologically stratified space X. Moreover, we explain how Ext-quivers relative to dual perversities behave under Verdier duality.

The Ext-quiver $\mathrm{Q}_p(X)$ of the category ${}^p\mathbf{Perv}(X)$, see Definition 2.2.2.6, has a vertex for each (isomorphism class of) simple perverse sheaf $\mathcal{S}_{\mathcal{L}} \in {}^p\mathbf{Perv}(X)$, that is a vertex for each irreducible local system $\mathcal{L} \in \mathbf{Loc}(S)$ on a stratum $S \subset X$. Note that, since we are considering topologically stratified spaces X with finitely many strata S each with finite fundamental group, the quiver $\mathrm{Q}_p(X)$ has finitely many vertices.

The number of arrows between two vertices labelled by $\mathcal{L}$ and $\mathcal{M}$ in $\mathrm{Q}_p(X)$ is given by the dimension of the vector space $\mathrm{Ext}^1_{{}^p\mathbf{Perv}(X)}(\mathcal{S}_{\mathcal{L}}, \mathcal{S}_{\mathcal{M}})^\vee$, that is there are $\dim \mathrm{Ext}^1(\mathcal{S}_{\mathcal{L}}, \mathcal{S}_{\mathcal{M}})$ arrows from the vertex which corresponds to the irreducible local system $\mathcal{L}$ to the one that corresponds to the irreducible local system $\mathcal{M}$. Since we can write the simple objects explicitly depending only on from where they arise, see Section 2.3.6 a) and c), the number of arrows in the quiver $\mathrm{Q}_p(X)$ can be calculated topologically. In Section 4.3, we will give a purely topological interpretation in terms of intersection cohomology of links of Ext-groups between simple objects.

We start with a Lemma which determines some cases in which there are no arrows between two vertices of the quiver $\mathrm{Q}_p(X)$.

Lemma 4.1.1.1. *Let X be a topologically stratified space. Let us consider two strata $S, T \subset X$ such that neither $S \leq T$ nor $T \geq S$ under the order of Remark 2.3.1.3. Then*

$$\mathrm{Ext}^1_{{}^p\mathbf{Perv}(X)}(\mathcal{S}_{\mathcal{L}}, \mathcal{S}_{\mathcal{M}}) = 0$$

for any $\mathcal{L} \in \mathbf{Loc}(S)$ and $\mathcal{M} \in \mathbf{Loc}(T)$.

Proof. Let $Z = \overline{S} \cap \overline{T}$ and define complementary inclusions $i : Z \hookrightarrow X$ and $j : U = X \smallsetminus Z \hookrightarrow X$. By Lemma 2.3.5.10 there is an inclusion

$$\mathrm{Ext}^1_{{}^p\mathbf{Perv}(X)}(\mathcal{S}_{\mathcal{L}}, \mathcal{S}_{\mathcal{M}}) \hookrightarrow \mathrm{Ext}^1_{{}^p\mathbf{Perv}(U)}(j^*\mathcal{S}_{\mathcal{L}}, j^*\mathcal{S}_{\mathcal{M}}).$$

Since the objects $j^*\mathcal{S}_{\mathcal{L}}$ and $j^*\mathcal{S}_{\mathcal{M}}$ are supported on disjoint closed subsets of U the right hand side vanishes. Therefore the claim follows. □

Verdier duality, see Theorem 2.3.4.5, sends p-perverse sheaves to p^*-perverse sheaves, where p and p^* is a pair of dual perversities. Therefore, the quivers $\mathrm{Q}_p(X)$ and $\mathrm{Q}_{p^*}(X)$ are related in the following way.

Lemma 4.1.1.2. *Let p and p^* be a pair of dual perversities on a topologically stratified space X. The quivers $\mathrm{Q}_p(X)$ and $\mathrm{Q}_{p^*}(X)$ for ${}^p\mathbf{Perv}(X)$ and ${}^{p^*}\mathbf{Perv}(X)$ are dual to each other.*

Proof. First note that the number of vertices of $\mathrm{Q}_p(X)$ and $\mathrm{Q}_{p^*}(X)$ is the same, since each irreducible local system $\mathcal{L} \in \mathbf{Loc}(S)$ on a stratum $S \subset X$ is sent by Verdier duality to its dual $\mathcal{L}^\vee \cong \mathcal{D}\mathcal{L} = \mathrm{Hom}(\mathcal{L}, \Bbbk)$, see Section 2.3.4. Let us consider two irreducible local systems $\mathcal{L}, \mathcal{M} \in \mathbf{Loc}(X)$, since we have

$$\begin{aligned}\mathrm{Ext}^1_{{}^{p^*}\mathbf{Perv}(X)}({}^{p^*}\mathcal{S}_{\mathcal{L}^\vee}, {}^{p^*}\mathcal{S}_{\mathcal{M}^\vee}) &\cong \mathrm{Ext}^1_{{}^p\mathbf{Perv}(X)}(\mathcal{D}^{p^*}\mathcal{S}_{\mathcal{M}^\vee}, \mathcal{D}^{p^*}\mathcal{S}_{\mathcal{L}^\vee}) \\ &\cong \mathrm{Ext}^1_{{}^p\mathbf{Perv}(X)}({}^p\mathcal{S}_{\mathcal{M}}, {}^p\mathcal{S}_{\mathcal{L}})\end{aligned}$$

the claims follows. □

Remark 4.1.1.3. *Lemma 4.1.1.2 says that one can obtain the quiver $\mathrm{Q}_{p^*}(X)$ for the category ${}^{p^*}\mathbf{Perv}(X)$ from the quiver $\mathrm{Q}_p(X)$ by relabelling the vertices, that is by assigning the dual local system $\mathcal{L}^\vee = \mathrm{Hom}(\mathcal{L}, \Bbbk)$ to the vertex labelled by $\mathcal{L}$ in $\mathrm{Q}_p(X)$, and reversing all the arrows.*

The next observation shows that the quiver $\mathrm{Q}_p(X)$ has no loops.

Proposition 4.1.1.4. *The* Ext-*quiver $\mathrm{Q}_p(X)$ of the category ${}^p\mathbf{Perv}(X)$ has no loops.*

Proof. Since the vertices of the quiver $\mathrm{Q}_p(X)$ are in one to one correspondence with simple objects in ${}^p\mathbf{Perv}(X)$, there is a loop at a vertex labelled with $\mathcal{L}$ for some $\mathcal{L} \in \mathbf{Loc}(S)$

where $S \subset X$ is a stratum if and only if $\mathrm{Ext}^1(\mathcal{S}_\mathcal{L}, \mathcal{S}_\mathcal{L}) \neq 0$. Let $i_S : S \hookrightarrow X$ be the inclusion of a stratum into X, then by Section 2.3.6.a) simple objects in ${}^p\mathbf{Perv}(X)$ are of the form ${}^p i_{S!*}\mathcal{L}[-p(S)]$. Therefore

$$\mathrm{Ext}^1(\mathcal{S}_\mathcal{L}, \mathcal{S}_\mathcal{L}) \cong \mathrm{Ext}^1({}^p i_{S!*}\mathcal{L}[-p(S)], {}^p i_{S!*}\mathcal{L}[-p(S)]) \cong \mathrm{Ext}^1_{{}^p\mathbf{Perv}(S)}(\mathcal{L}, \mathcal{L}) = 0$$

since the intermediate extension functor is fully faithful, see Lemma 2.3.5.10, and ${}^p\mathbf{Perv}(S)$ is semisimple if $\pi_1(S)$ is finite and $\mathrm{char}(\Bbbk)$ does not divide the order of $\pi_1(S)$. $\square$

4.1.2 Examples

In this section we calculate the Ext-quiver $\mathrm{Q}_p(X)$ in some particular cases. We start with $X = \mathbb{P}^1$ stratified by a point and its open complement as considered in Example 2.3.1.6.iii). We determine $\mathrm{Q}_p(X)$ for the three meaningful perversities, namely middle, zero and top perversity. We are able to do this by describing the two simple objects corresponding to the two strata as sheaves on X. Moreover, we note that Verdier duality exchanges the quivers relative to a pair of dual perversities. For the case of the zero perversity, we also determine the Ext-quiver for the case of the first quadrant stratified by one axis and the origin and $X = S^1$ stratified by a point and its open complement. We observe that the case of the zero perversity can be equivalently described as constructible sheaves and representations of the exit path category.

Example 4.1.2.1. *Let us consider $X = \mathbb{P}^1$ stratified as in Example 2.3.1.6.iii), that is there are two strata $Z \cong \{\mathrm{pt}\}$ and $U \cong \mathbb{C}$. We want to determine the quivers $\mathrm{Q}_p(X)$ for the middle, zero and top perversities.*

i) For the middle perversity $m(S) = (m(U), m(Z)) = (-1, 0)$ the two simple objects in ${}^m\mathbf{Perv}(X)$ are

$$
\begin{aligned}
{}^m\mathcal{S}_U &\cong j_{!*}\Bbbk_U[1] \cong \Bbbk_X[1] \\
{}^m\mathcal{S}_Z &\cong i_*\Bbbk_Z.
\end{aligned}
$$

Then, the Ext^1*-groups between the simple objects are*

$$\begin{aligned}\mathrm{Ext}^1_{{}^m\mathbf{Perv}(X)}({}^m\mathcal{S}_U, {}^m\mathcal{S}_Z) &\cong \mathrm{Ext}^1_{{}^m\mathbf{Perv}(X)}(\Bbbk_X[1], i_*\Bbbk_Z)\\ &\cong \mathrm{Ext}^1_{{}^m\mathbf{Perv}(Z)}(i^*\Bbbk_X[1], \Bbbk_Z)\\ &\cong \mathrm{Hom}(\Bbbk_Z, \Bbbk_Z) \cong \Bbbk,\end{aligned}$$

while

$$\begin{aligned}\mathrm{Ext}^1_{{}^m\mathbf{Perv}(X)}({}^m\mathcal{S}_Z, {}^m\mathcal{S}_U) &\cong \mathrm{Ext}^1_{{}^m\mathbf{Perv}(X)}(i_*\Bbbk_Z, \Bbbk_X[1])\\ &\cong \mathrm{Ext}^1_{{}^m\mathbf{Perv}(Z)}(\Bbbk_Z, i^!\Bbbk_X[1])\\ &\cong \mathrm{Ext}^1_{{}^m\mathbf{Perv}(Z)}(\Bbbk_Z, i^*\Bbbk_X[-1])\\ &\cong \mathrm{Hom}(\Bbbk_Z, \Bbbk_Z) \cong \Bbbk\end{aligned}$$

since $i^!\Bbbk_X \cong i^*\Bbbk_X[-2]$*, see [GM83, Section 1.13.(5)]. Therefore, the* Ext*-quiver for the category* ${}^m\mathbf{Perv}(X)$ *is given by*

$$\mathrm{Q}_m(X) = \begin{tikzcd} \Bbbk_U \arrow[r, bend left, "\alpha"] & \Bbbk_Z \arrow[l, bend left, "\beta"] \end{tikzcd}. \tag{4.3}$$

ii) For the zero perversity $o(S) = (o(U), o(Z)) = (0, 0)$ *the two simple objects are*

$$\begin{aligned}{}^o\mathcal{S}_U &\cong {}^o j_{!*}\Bbbk_U \cong j_!\Bbbk_U\\ {}^o\mathcal{S}_Z &\cong i_*\Bbbk_Z.\end{aligned}$$

Then, the Ext^1*-groups between the simple objects are*

$$\mathrm{Ext}^1_{{}^o\mathbf{Perv}(X)}({}^o\mathcal{S}_U, {}^o\mathcal{S}_Z) \cong \mathrm{Ext}^1_{{}^o\mathbf{Perv}(X)}(j_!\Bbbk_U, i_*\Bbbk_Z) \cong 0$$

by adjunction, while

$$\begin{aligned}\mathrm{Ext}^1_{{}^o\mathbf{Perv}(X)}({}^0\mathcal{S}_Z, {}^o\mathcal{S}_U) &\cong \mathrm{Ext}^1_{{}^o\mathbf{Perv}(X)}(i_*\Bbbk_Z, j_!\Bbbk_U)\\ &\cong \mathrm{Ext}^1_{{}^o\mathbf{Perv}(Z)}(\Bbbk_Z, i^!j_!\Bbbk_U)\\ &\cong \mathrm{Ext}^1_{{}^o\mathbf{Perv}(Z)}(\Bbbk_Z, i^*j_*\Bbbk_U[-1])\\ &\cong \mathrm{Hom}(\Bbbk_Z, \Bbbk_Z) \cong \Bbbk\end{aligned}$$

since $i^! j_! = i^* j_*[-1]$, *see Lemma 2.3.2.7, and using Lemma 2.3.2.8. Therefore, the* Ext-*quiver for* ${}^o\mathbf{Perv}(X)$ *is*

$$\mathrm{Q}_o(X) = \; \Bbbk_U \xleftarrow{\beta} \Bbbk_Z \; .$$

iii) For the top perversity $t(S) = (t(U), t(Z)) = (-2, 0)$ *the two simple objects are*

$$\begin{aligned} {}^t\mathcal{S}_U &\cong {}^t j_{!*}\Bbbk_U[2] \cong j_*\Bbbk_U[2] \\ {}^t\mathcal{S}_Z &\cong i_*\Bbbk_Z. \end{aligned}$$

Then, the Ext^1*-groups between the simple objects are*

$$\begin{aligned} \mathrm{Ext}^1_{{}^t\mathbf{Perv}(X)}({}^t\mathcal{S}_U, {}^t\mathcal{S}_Z) &\cong \mathrm{Ext}^1_{{}^t\mathbf{Perv}(X)}(j_*\Bbbk_U[2], i_*\Bbbk_Z) \\ &\cong \mathrm{Ext}^1_{{}^t\mathbf{Perv}(Z)}(i^*j_*\Bbbk_U[2], \Bbbk_Z) \\ &\cong \mathrm{Hom}(i^*j_*\Bbbk_U[1], \Bbbk_Z) \\ &\cong \mathrm{Hom}(\Bbbk_Z[1] \oplus \Bbbk_Z, \Bbbk_Z) \\ &\cong \mathrm{Hom}(\Bbbk_Z, \Bbbk_Z) \cong \Bbbk \end{aligned}$$

as by Lemma 2.3.2.8 we have $i^*j_*\Bbbk_U \cong \Bbbk_Z \oplus \Bbbk_Z[-1]$. *On the other hand,*

$$\mathrm{Ext}^1_{{}^t\mathbf{Perv}(X)}({}^t\mathcal{S}_Z, {}^t\mathcal{S}_U) \cong \mathrm{Ext}^1_{{}^t\mathbf{Perv}(X)}(i_*\Bbbk_Z, j_*\Bbbk_U[2]) \cong 0$$

by adjunction. Therefore, the Ext-*quiver for* ${}^t\mathbf{Perv}(X)$ *is*

$$\mathrm{Q}_t(X) = \; \Bbbk_U \xrightarrow{\alpha} \Bbbk_Z \; .$$

Remark 4.1.2.2. *Note that as predicted by Lemma 4.1.1.2 the* Ext-*quiver* $\mathrm{Q}_o(X)$ *of Example 4.1.2.1.ii) and* Ext-*quiver* $\mathrm{Q}_t(X)$ *of Example 4.1.2.1.iii) are dual to each other, while the one in Example 4.1.2.1.i) is self-dual.*

Remark 4.1.2.3. *Following an unpublished observation of MacPherson, see [Tre09, Section 1.1], for a fixed stratification* $\mathscr{S}$ *of a topologically stratified space* X, *one can define the* **exit path category**, *denoted by* $\mathrm{EP}_{\leq 1}(X, \mathscr{S})$. *Its objects are points of* X *and its morphisms are homotopy classes of exit paths, that is homotopy classes of paths in* X *which are allowed to exit a stratum only to go to a higher dimensional one. In particular, [Tre09,* *Theorem 1.2] gives an equivalence*

$$\mathbf{Constr}(X) \simeq \mathrm{Fun}(\mathrm{EP}_{\leq 1}(X, \mathscr{S}), \mathbf{Vect}_{\Bbbk}).$$

Therefore, in order to calculate the Ext*-quiver* $\mathrm{Q}_o(X)$ *for the category* ${}^o\mathbf{Perv}(X) \simeq \mathbf{Constr}(X)$, *we can consider the exit path category instead.*

Example 4.1.2.4. *The exit path category of* $X = \mathbb{P}^1$ *with stratification given by a point* $\{\infty\}$ *and its open complement* $\mathbb{C}$ *is equivalent to a category with two objects,* $\{\infty\}$ *and* $0 \in \mathbb{C}$ *with the only non trivial arrow the exit path from* $\{\infty\}$ *to* 0. *That is,* $\mathrm{Fun}(\mathrm{EP}_{\leq 1}(\mathbb{P}^1, \mathscr{S}), \mathbf{Vect}_{\Bbbk})$ *is equivalent to*

$$0 \longleftarrow \{\infty\} \ ,$$

see [Tre09, Example 1.4]. This quiver is isomorphic to the one of Example 4.1.2.1.ii), as anticipated by Remark 4.1.2.3.

Example 4.1.2.5. *Let us consider the first quadrant* $X = (\mathbb{R}_{\geq 0})^2$ *stratified by one axis and the origin as in Example 2.3.1.6.i). There are three strata,*

$$\begin{aligned} S_0 &= X_0 \cong \{0\}, \\ S_1 &= X_1 \smallsetminus X_0 \cong \mathbb{R}_{>0}, \\ S_2 &= X_2 \smallsetminus X_1 \cong \mathbb{R}_{>0} \times \mathbb{R}_{\geq 0}. \end{aligned}$$

We will use the following maps

$$\begin{tikzcd} S_2 \arrow[r, hook, "j"] & X & Z = S_1 \cup S_0 \arrow[l, hook', "i"'] & S_0 \arrow[l, hook', "k"'] \\ & & S_1 \arrow[u, hook, "l"] & \end{tikzcd}$$

where j *and* l *are open while* i *and* k *are closed. Let us consider the zero perversity, that is*

$$o(S) = (o(S_2), o(S_1), o(S_0)) = (0, 0, 0).$$

We want to determine the quiver $\mathrm{Q}_o(X)$ *for the category* ${}^o\mathbf{Perv}((\mathbb{R}_{\geq 0})^2) \simeq \mathbf{Constr}((\mathbb{R}_{\geq 0})^2)$. *The simple objects, see Example 2.3.7.3, are given by extensions by zero of constant sheaves* *on each stratum, that is*

$$\mathcal{S}_2 \cong j_! \Bbbk_{S_2}, \qquad \mathcal{S}_1 \cong i_* l_! \Bbbk_{S_1} \qquad \text{and} \qquad \mathcal{S}_0 \cong i_* k_* \Bbbk_{S_0}.$$

In order to find the quiver $\mathrm{Q}_o(X)$*, it is enough to calculate the* Ext^1*-groups between simple objects. By Proposition 4.1.1.4, we have that* $\mathrm{Ext}^1(\mathcal{S}_i, \mathcal{S}_i) \cong 0$ *for* $i = 0, 1, 2$*. Moreover, we have*

$$\begin{aligned}
\mathrm{Ext}^1(\mathcal{S}_0, \mathcal{S}_1) &\cong \mathrm{Ext}^1(i_* k_* \Bbbk_{S_0}, i_* l_! \Bbbk_{S_1}) \cong \mathrm{Ext}^1_{\mathbf{D}_c(Z)}(k_* \Bbbk_{S_0}, l_! \Bbbk_{S_1}) \cong \mathrm{Ext}^1_{\mathbf{D}_c(S_0)}(\Bbbk_{S_0}, k^! l_! \Bbbk_{S_1}) \\
&\cong \mathrm{Ext}^1_{\mathbf{D}_c(S_0)}(\Bbbk_{S_0}, k^* l_* \Bbbk_{S_1}[-1]) \cong \mathrm{Hom}(\Bbbk_{S_0}, \Bbbk_{S_0}) \cong \Bbbk, \\
\mathrm{Ext}^1(\mathcal{S}_0, \mathcal{S}_2) &\cong \mathrm{Ext}^1(i_* k_* \Bbbk_{S_0}, j_! \Bbbk_{S_2}) \cong \mathrm{Ext}^1_{\mathbf{D}_c(Z)}(k_* \Bbbk_{S_0}, i^! j_! \Bbbk_{S_2}) \cong \mathrm{Ext}^1_{\mathbf{D}_c(Z)}(k_* \Bbbk_{S_0}, i^* j_* \Bbbk_{S_2}[-1]) \\
&\cong \mathrm{Hom}(k_* \Bbbk_{S_0}, i^* j_* \Bbbk_{S_2}) \cong 0, \\
\mathrm{Ext}^1(\mathcal{S}_1, \mathcal{S}_0) &\cong \mathrm{Ext}^1(i_* l_! \Bbbk_{S_1}, i_* k_* \Bbbk_{S_0}) \cong \mathrm{Ext}^1_{\mathbf{D}_c(Z)}(l_! \Bbbk_{S_1}, k_* \Bbbk_{S_0}) \cong 0, \\
\mathrm{Ext}^1(\mathcal{S}_1, \mathcal{S}_2) &\cong \mathrm{Ext}^1(i_* l_! \Bbbk_{S_1}, j_! \Bbbk_{S_2}) \cong \mathrm{Ext}^1_{\mathbf{D}_c(Z)}(l_! \Bbbk_{S_1}, i^! j_! \Bbbk_{S_2}) \cong \mathrm{Ext}^1_{\mathbf{D}_c(Z)}(l_! \Bbbk_{S_1}, i^* j_* \Bbbk_{S_2}[-1]) \\
&\cong \mathrm{Hom}(l_! \Bbbk_{S_1}, i^* j_* \Bbbk_{S_2}) \cong \mathrm{Hom}(l_! \Bbbk_{S_1}, i^* j_* \Bbbk_{S_2}) \cong \Bbbk, \\
\mathrm{Ext}^1(\mathcal{S}_2, \mathcal{S}_0) &\cong \mathrm{Ext}^1(j_! \Bbbk_{S_2}, i_* k_* \Bbbk_{S_1}) \cong 0, \\
\mathrm{Ext}^1(\mathcal{S}_2, \mathcal{S}_1) &\cong \mathrm{Ext}^1(j_! \Bbbk_{S_2}, i_* l_! \Bbbk_{S_1}) \cong 0.
\end{aligned}$$

Therefore, the Ext*-quiver of* ${}^o\mathbf{Perv}((\mathbb{R}_{\geq 0})^2) \simeq \mathbf{Constr}((\mathbb{R}_{\geq 0})^2)$ *is*

$$\mathrm{Q}_o(X) = \ \Bbbk_{S_2} \longleftarrow \Bbbk_{S_1} \longleftarrow \Bbbk_{S_0} \ .$$

As noted in Remark 4.1.2.3, this can also be obtained from the exit path category point of view.

Example 4.1.2.6. *Let us consider* $X = S^1$ *stratified by a point and its open complement as in Example 2.3.1.6.iv). The two strata are* $S_0 \cong \{0\}$ *and* $S_1 \cong (-1, 1)$ *and we have complementary open and closed inclusions* $S_1 \overset{j}{\hookrightarrow} X \overset{i}{\hookleftarrow} S_0$*. Let us consider the zero perversity, that is*

$$o(S) = (o(S_1), o(S_0)) = (0, 0).$$

We want to determine the quiver $\mathrm{Q}_o(X)$ *for the category* ${}^o\mathbf{Perv}(S^1) \simeq \mathbf{Constr}(S^1)$*. The simple objects, see Example 2.3.7.3, are given by extensions by zero of constant sheaves on* *each stratum, that is*

$$\mathcal{S}_1 \cong j_! \Bbbk_{S_1} \qquad \textit{and} \qquad \mathcal{S}_0 \cong i_* \Bbbk_0.$$

In order to find the quiver $\mathrm{Q}_o(X)$*, it is enough to calculate the* Ext^1*-groups between simple objects. By Proposition 4.1.1.4, we have that* $\mathrm{Ext}^1(\mathcal{S}_i, \mathcal{S}_i) \cong 0$ *for* $i = 0, 1$*. Moreover, we have*

$$\begin{aligned} \mathrm{Ext}^1(\mathcal{S}_1, \mathcal{S}_1) &\cong \mathrm{Ext}^1(j_! \Bbbk_{S_1}, i_* \Bbbk_0) \cong 0 \quad \textit{by adjunction}, \\ \mathrm{Ext}^1(\mathcal{S}_0, \mathcal{S}_1) &\cong \mathrm{Ext}^1(i_* \Bbbk_0, j_! \Bbbk_{S_1}) \cong \mathrm{Ext}^1(\Bbbk_0, i^! j_! \Bbbk_{S_1}) \quad \textit{by Lemma 2.3.2.7} \\ &\cong \mathrm{Ext}^1(\Bbbk_0, i^* j_* \Bbbk_{S_1}[-1]) \cong \mathrm{Hom}(\Bbbk_0, \Bbbk_0 \oplus \Bbbk_0) \cong \Bbbk^2, \end{aligned}$$

since by Lemma 2.3.2.8 we have $L_{S_0 \subset S_1} \cong (-1, 0) \cup (0, 1)$*. Therefore, the* Ext*-quiver of* ${}^o\mathbf{Perv}(S^1) \simeq \mathbf{Constr}(S^1)$ *is*

$$\mathrm{Q}_o(S^1) = \begin{tikzcd} \Bbbk_{S_1} & \Bbbk_{S_0} \arrow[l, bend right] \arrow[l, bend left] \end{tikzcd} .$$

As noted in Remark 4.1.2.3, this can also be obtained from the exit path category point of view. One can note that the Ext*-quiver* $\mathrm{Q}_o(S^1)$ *is isomorphic to the Kronecker quiver. It is a well-known fact that the path algebra of the* Ext*-quiver* $\mathrm{Q}_o(S^1)$ *is of infinite representation type.*

4.2 Relations $\mathrm{I}_p(X)$ of the Ext-quiver $\mathrm{Q}_p(X)$

In this section we explain how to determine the ideal of relations $\mathrm{I}_p(X)$ of the Ext-quiver $\mathrm{Q}_p(X)$ for the category ${}^p\mathbf{Perv}(X)$. In general, this turns out to be a very difficult task. The main reason behind that is that the information needed to determine the ideal of relations $\mathrm{I}_p(X)$ is not encoded in the structure of triangulated category of $\mathbf{D}_c(X)$, but in an A_∞-structure which is much more complicated. Nevertheless, we can explain how to obtain the quadratic part of relations using an inductive process as this case only uses the structure of the constructible derived category $\mathbf{D}_c(X)$. Note that this completely characterises the case of quadratic algebras, which for instance include Koszul algebras.

The 'easiest' approach to determine the Ext-quiver $\mathrm{Q}_p(X)$ and the ideal of relations $\mathrm{I}_p(X)$ is to compute the Ext-algebra, that is one should calculate the groups $\mathrm{Ext}^i_{\mathbf{D}_c(X)}(\mathcal{S}_\mathcal{L}, \mathcal{S}_\mathcal{M})$ for every local system $\mathcal{L}$ and $\mathcal{M}$ and $i \geq 0$. In the case that the Ext-algebra is quadratic, then we are able to determine the quiver and relations. Indeed, the Ext^1-groups give the number of arrows between vertices, while the Ext^2-groups give the relations starting and ending at the corresponding vertices. In the case that the Ext-algebra is not quadratic, we can tell if there are non-quadratic relations, but in general we cannot conclude more than that.

4.2.1 A_∞-structure on the Ext-Algebra

Let us denote by $\{\mathcal{S}_\mathcal{L}\}$ the set of representatives of isomorphism classes of simple objects in ${}^p\mathbf{Perv}(X)$, where $\mathcal{L} \in \mathbf{Loc}(S)$ for some stratum $S \subset X$. Note that this set is finite since we are considering topologically stratified spaces with finitely many strata, each with finite fundamental group. Moreover, let us denote by $\mathcal{S} = \oplus_\mathcal{L}\mathcal{S}_\mathcal{L}$ the (finite) sum of all (isomorphism classes of) simple objects in ${}^p\mathbf{Perv}(X)$. Furthermore, let $\{\mathcal{P}_\mathcal{L}\}$ be the finite set of projective covers of simple objects in ${}^p\mathbf{Perv}(X)$, built by using the technique of Sections 3.3.1 and 3.3.2. Then, by Remark 3.4.0.3, the object $\mathcal{P} = \oplus_\mathcal{L}\mathcal{P}_\mathcal{L}$ is a projective generator of the category ${}^p\mathbf{Perv}(X)$. Recall that, as explained in the proof of Proposition 3.4.0.5, the choice of a basis for each $\mathrm{Ext}^1_{{}^p\mathbf{Perv}(X)}(\mathcal{S}_\mathcal{L}, \mathcal{S}_\mathcal{M})$ determines a unique algebra homomorphism $\Bbbk\mathrm{Q}_p(X) \twoheadrightarrow \mathrm{End}(\mathcal{P})$ and therefore an ideal $\mathrm{I}_p(X)$ such that $\mathrm{End}(\mathcal{P}) \cong \Bbbk\mathrm{Q}_p(X)/\mathrm{I}_p(X)$. In principle, the ideal $\mathrm{I}_p(X)$ of relations can be computed by taking projective resolutions of simple objects in ${}^p\mathbf{Perv}(X)$. This induces an A_∞-structure on the Ext-algebra $\mathrm{Ext}^*_{{}^p\mathbf{Perv}(X)}(\mathcal{S}, \mathcal{S})$, which in practice can be very difficult to handle.

4.2.2 Quadratic Algebras

We now recall some basic definitions and fact about quadratic algebras, see [BGS96, Section 1 and 2] (where actually these definitions are given for rings). This class of algebras will be of particular interest because we can completely determine both the Ext-quiver $\mathrm{Q}_p(X)$ and its relations $\mathrm{I}_p(X)$.

In Definition 2.2.1.1 we introduced the notion of algebra over a field $\Bbbk$. We now introduce algebras which also have a grading.

We consider a **(positively) graded $\Bbbk$-algebra** $\mathrm{A} = \bigoplus_{i \geq 0} \mathrm{A}_i$ satisfying the following conditions:

- $\mathrm{A}_i\mathrm{A}_j \subseteq \mathrm{A}_{i+j}$ for any i, j;
- $\mathrm{A}_0 \cong \Bbbk \times \ldots \times \Bbbk$;

- A_1 is a finite dimensional $\Bbbk$-vector space.

The elements of A_i are called homogeneous of degree i while A_i is the degree i component of A.

Throughout this section, let $V \in \mathbf{Vect}_\Bbbk$ denote a finite dimensional vector space over a field $\Bbbk$.

Definition 4.2.2.1. *The* **tensor algebra** *over the vector space V is*

$$\mathrm{T}V = \Bbbk 1 \oplus V \oplus V^{\otimes 2} \oplus V^{\otimes 3} \oplus \dots$$

A special role is played by a class of algebras which arise as particular quotients of the tensor algebra over a vector space V.

Definition 4.2.2.2. *Let $\mathrm{R} \subseteq V^{\otimes 2}$ and denote by $\langle \mathrm{R} \rangle$ the ideal generated by R. The algebra*

$$\mathrm{A(V,R)} = \mathrm{T}V/\langle \mathrm{R} \rangle$$

is the **quadratic algebra** *with $\langle \mathrm{R} \rangle$ as ideal of relations.*

Remark 4.2.2.3. *The algebra $\mathrm{A(V,R)}$ inherits a grading given by the length of 'words', that is*

$$\mathrm{A(V,R)} = \Bbbk 1 \oplus V \oplus \left(V^{\otimes 2}/\mathrm{R}\right) \oplus \dots \oplus \left(V^{\otimes n} / \sum_{i+2+j=n} V^{\otimes i} \otimes \mathrm{R} \otimes V^{\otimes j} \right) \oplus \dots$$

We now give the definition of Koszul algebra.

Definition 4.2.2.4. *A (graded) algebra A is* **Koszul** *if each simple A-module $M \in A$-***mod** *admits a graded projective resolution of the form*

$$\dots \to P_{n+1} \to P_n \to \dots \to P_1 \to P_0 \to M \to 0$$

where P_j is projective and generated by a set of homogeneous elements of degree j.

Proposition 4.2.2.5. *[BGS96, Proposition 1.2.3] Any Koszul algebra is quadratic.*

Example 4.2.2.6. *In the situation of Example 4.1.2.1.i)-iii), one can extend the calculation done for the Ext^1-groups and calculate the Ext-algebra. It turns out that in all three examples the Ext-algebra is quadratic.*

i) Let us consider the middle perversity on $X = \mathbb{P}^1$*. We have*

$$\operatorname{Ext}^i_{\mathbf{D}_c(X)}({}^m\mathcal{S}_U, {}^m\mathcal{S}_U) \cong \operatorname{Ext}^i_{\mathbf{D}_c(X)}(\Bbbk_X[1], \Bbbk_X[1]) \cong H^i(X;\Bbbk) \cong \begin{cases} \Bbbk & \text{if } i = 0,2 \\ 0 & \text{otherwise} \end{cases},$$

$$\begin{aligned}\operatorname{Ext}^i_{\mathbf{D}_c(X)}({}^m\mathcal{S}_U, {}^m\mathcal{S}_Z) &\cong \operatorname{Ext}^i_{\mathbf{D}_c(X)}(\Bbbk_X[1], i_*\Bbbk_Z) \cong \operatorname{Ext}^i_{\mathbf{D}_c(Z)}(i^*\Bbbk_X[1], \Bbbk_Z) \\ &\cong \operatorname{Ext}^i_{\mathbf{D}_c(Z)}(\Bbbk_Z[1], \Bbbk_Z) \cong \begin{cases} \Bbbk & \text{if } i = 1 \\ 0 & \text{if } i \neq 1 \end{cases},\end{aligned}$$

$$\begin{aligned}\operatorname{Ext}^i_{\mathbf{D}_c(X)}({}^m\mathcal{S}_Z, {}^m\mathcal{S}_U) &\cong \operatorname{Ext}^i_{\mathbf{D}_c(X)}(i_*\Bbbk_Z, \Bbbk_X[1]) \cong \operatorname{Ext}^i_{\mathbf{D}_c(Z)}(\Bbbk_Z, i^!\Bbbk_X[1]) \\ &\cong \operatorname{Ext}^i_{\mathbf{D}_c(Z)}(\Bbbk_Z, \Bbbk_Z[-1]) \cong \begin{cases} \Bbbk & \text{if } i = 1 \\ 0 & \text{if } i \neq 1 \end{cases},\end{aligned}$$

$$\operatorname{Ext}^i_{\mathbf{D}_c(X)}({}^m\mathcal{S}_Z, {}^m\mathcal{S}_Z) \cong \operatorname{Ext}^i_{\mathbf{D}_c(X)}(i_*\Bbbk_Z, i_*\Bbbk_Z) \cong H^i(Z;\Bbbk) \cong \begin{cases} \Bbbk & \text{if } i = 0 \\ 0 & \text{otherwise} \end{cases}.$$

Hence, the relations of the Ext*-quiver* $\mathrm{Q}_m(X)$ *are given by* $\mathrm{I}_m(X) = \langle \beta \circ \alpha \rangle$ *and* $\mathrm{A}_m(X)$ *is a matrix algebra.*

ii) Let us consider the zero perversity on $X = \mathbb{P}^1$*. We have*

$$\operatorname{Ext}^i_{\mathbf{D}_c(X)}({}^o\mathcal{S}_U, {}^o\mathcal{S}_U) \cong \operatorname{Ext}^i_{\mathbf{D}_c(X)}(j_!\Bbbk_U, j_!\Bbbk_U) \cong H^i(U;\Bbbk) \cong \begin{cases} \Bbbk & \text{if } i = 0 \\ 0 & \text{otherwise} \end{cases},$$

$$\operatorname{Ext}^i_{\mathbf{D}_c(X)}({}^o\mathcal{S}_U, {}^o\mathcal{S}_Z) \cong \operatorname{Ext}^i_{\mathbf{D}_c(X)}(j_!\Bbbk_U, i_*\Bbbk_Z) \cong 0,$$

$$\begin{aligned}\operatorname{Ext}^i_{\mathbf{D}_c(X)}({}^o\mathcal{S}_Z, {}^o\mathcal{S}_U) &\cong \operatorname{Ext}^i_{\mathbf{D}_c(X)}(i_*\Bbbk_Z, j_!\Bbbk_U) \cong \operatorname{Ext}^i_{\mathbf{D}_c(Z)}(\Bbbk_Z, i^!j_!\Bbbk_U) \\ &\cong \operatorname{Ext}^i_{\mathbf{D}_c(Z)}(\Bbbk_Z, i^*j_*\Bbbk_U[-1]) \cong \begin{cases} \Bbbk & \text{if } i = 1 \\ 0 & \text{if } i \neq 1 \end{cases},\end{aligned}$$

$$\operatorname{Ext}^i_{\mathbf{D}_c(X)}({}^o\mathcal{S}_Z, {}^o\mathcal{S}_Z) \cong \operatorname{Ext}^i_{\mathbf{D}_c(X)}({}^m\mathcal{S}_Z, {}^m\mathcal{S}_Z) \cong \begin{cases} \Bbbk & \text{if } i = 0 \\ 0 & \text{otherwise} \end{cases}.$$

Hence, as all the Ext^2*-groups vanish there are no relations, that is* $\mathrm{I}_o(X) = 0$ *and* $\mathrm{A}_o(X)$ *is a three dimensional matrix algebra (the path algebra of the* $\mathbb{A}_2$ *quiver).*

iii) The case of the top perversity is completely dual to the zero perversity case, that is *we have* $\mathrm{I}_t(X) = 0$ *and* $\mathrm{A}_t(X) \cong \Bbbk\mathrm{Q}_t(X) \cong \mathrm{A}_o(X)^{op}$.

Example 4.2.2.7. *In the situation of Example 4.1.2.5 one can calculate the* Ext-*algebra and check that*

$$\mathrm{Ext}^k_{\mathbf{D}_c(X)}(\mathcal{S}_i, \mathcal{S}_j) = 0 \quad \forall k \geq 2.$$

This implies that the Ext-*algebra is quadratic and, as all the* Ext^2-*groups vanish that there are no relations, that is* $\mathrm{I}_o(X) = 0$. *Note that, in order to conclude that the ideal of relations* $\mathrm{I}_o(X)$ *is zero, it is actually enough to check that* $\mathrm{Ext}^2_{\mathbf{D}_c(X)}(\mathcal{S}_0, \mathcal{S}_2) = 0$. *The same argument applies to the case of Example 4.1.2.6, that is* $\mathrm{I}_o(S^1) = 0$,

4.2.3 Quadratic Part of Relations

While in general in order to determine the ideal of relations $\mathrm{I}_p(X)$ of the Ext-quiver $\mathrm{Q}_p(X)$ one needs to deal with an A_∞-structure on the Ext-algebra, an easier goal is to find the quadratic part of the relations. Indeed, this can be achieved using only the structure of $\mathbf{D}_c(X)$. Let us denote by $\mathcal{S} = \oplus_i \mathcal{S}_i$ the (finite) sum of all (isomorphism classes of) simple objects in ${}^p\mathbf{Perv}(X)$. Moreover, let $S \subset X$ be a closed stratum and consider complementary maps

$$U \overset{j}{\hookrightarrow} X \overset{i}{\hookleftarrow} S.$$

We denote the path algebra of the Ext-quiver with relations $(\mathrm{Q}_p(X), \mathrm{I}_p(X))$ by $\mathrm{A}_p(X) \cong \Bbbk\mathrm{Q}_p(X)/\mathrm{I}_p(X)$ and by $\mathrm{Q}^k_p(X)$ the homogeneous length k paths. Therefore, we can define $\mathrm{A}^k_p(X) = \Bbbk\mathrm{Q}^k_p(X)/\mathrm{I}_p(X)$ and $\mathrm{A}^{\geq k}_p(X) = \oplus_{i \geq k} \Bbbk\mathrm{Q}^i_p(X)/\mathrm{I}_p(X)$. Note that $\mathrm{A}^1_p(X) \cong \mathrm{Q}^1_p(X)$.

Definition 4.2.3.1. *Let* $\mathrm{I}_p(X)$ *be the ideal of relations, the* **quadratic part of (the ideal of) relations** *is given by*

$$\bar{\mathrm{I}}_p(X) = \mathrm{im}(\mathrm{I}_p(X) \to \mathrm{A}_p(X)/\mathrm{A}^{\geq 3}_p(X)).$$

We now show that the map between Ext^2-groups in the heart ${}^p\mathbf{Perv}(X)$ and the Ext^2-groups in the ambient derived category $\mathbf{D}_c(X)$ is always a monomorphism.

Lemma 4.2.3.2. *Let* X *be a topologically stratified space with finitely many strata, each with finite fundamental group and suppose the characteristic of* $\Bbbk$ *does not divide the orders of the fundamental groups of strata. If for all* $\mathcal{E}, \mathcal{F} \in {}^p\mathbf{Perv}(X)$ *and* $k < i$ *the canonical* *maps* $\mathrm{Ext}^k_{{}^p\mathbf{Perv}(X)}(\mathcal{E},\mathcal{F}) \to \mathrm{Ext}^k_{\mathbf{D}_c(X)}(\mathcal{E},\mathcal{F})$ *are isomorphisms, then*

$$\mathrm{Ext}^i_{{}^p\mathbf{Perv}(X)}(\mathcal{E},\mathcal{F}) \to \mathrm{Ext}^i_{\mathbf{D}_c(X)}(\mathcal{E},\mathcal{F})$$

is a monomorphism. In particular, the canonical map

$$\mathrm{Ext}^2_{{}^p\mathbf{Perv}(X)}(\mathcal{E},\mathcal{F}) \to \mathrm{Ext}^2_{\mathbf{D}_c(X)}(\mathcal{E},\mathcal{F})$$

is always a monomorphism.

Proof. First note that $\left(\mathrm{Ext}^i_{{}^p\mathbf{Perv}(X)}(\mathcal{E},-)\right)_{i\in\mathbb{Z}_{\geq 0}}$ is an effaceable functor and therefore a universal δ-functor. That is, for any $\mathcal{E},\mathcal{F} \in {}^p\mathbf{Perv}(X)$ there are canonical maps $\mathrm{Ext}^i_{{}^p\mathbf{Perv}(X)}(\mathcal{E},\mathcal{F}) \to \mathrm{Ext}^i_{\mathbf{D}_c(X)}(\mathcal{E},\mathcal{F})$ as for a fixed $\mathcal{E}$ those are δ-functors from the category of p-perverse sheaves to abelian groups. The short exact sequence $0 \to \ker\pi \to \mathcal{P}_{\mathcal{E}} \xrightarrow{\pi} \mathcal{E} \to 0$ in ${}^p\mathbf{Perv}(X)$ induces the following commutative diagram

$$\begin{tikzcd}
\ldots \arrow[r] & \mathrm{Ext}^{i-1}_{{}^p\mathbf{Perv}(X)}(\mathcal{P}_{\mathcal{E}},\mathcal{F}) \arrow[r] \arrow[d] & \mathrm{Ext}^{i-1}_{{}^p\mathbf{Perv}(X)}(\ker\pi,\mathcal{F}) \arrow[r] \arrow[d] & \mathrm{Ext}^{i}_{{}^p\mathbf{Perv}(X)}(\mathcal{E},\mathcal{F}) \arrow[r] \arrow[d] & \mathrm{Ext}^{i}_{{}^p\mathbf{Perv}(X)}(\mathcal{P}_{\mathcal{E}},\mathcal{F}) \arrow[r] \arrow[d] & \ldots \\
\ldots \arrow[r] & \mathrm{Ext}^{i-1}_{\mathbf{D}_c(X)}(\mathcal{P}_{\mathcal{E}},\mathcal{F}) \arrow[r] & \mathrm{Ext}^{i-1}_{\mathbf{D}_c(X)}(\ker\pi,\mathcal{F}) \arrow[r] & \mathrm{Ext}^{i}_{\mathbf{D}_c(X)}(\mathcal{E},\mathcal{F}) \arrow[r] & \mathrm{Ext}^{i}_{\mathbf{D}_c(X)}(\mathcal{P}_{\mathcal{E}},\mathcal{F}) \arrow[r] & \ldots
\end{tikzcd} \tag{4.4}$$

Since $\mathcal{P}_{\mathcal{E}} \in {}^p\mathbf{Perv}(X)$ is projective the first and last term in the top row of (4.4) are zero, hence $\mathrm{Ext}^{i-1}_{{}^p\mathbf{Perv}(X)}(\ker\pi,\mathcal{F}) \cong \mathrm{Ext}^i_{{}^p\mathbf{Perv}(X)}(\mathcal{E},\mathcal{F})$. Moreover, by assumption the first two downwards maps on the left in (4.4) are isomorphisms which imply that $\mathrm{Ext}^{i-1}_{\mathbf{D}_c(X)}(\ker\pi,\mathcal{F}) \to \mathrm{Ext}^i_{\mathbf{D}_c(X)}(\mathcal{E},\mathcal{F})$ is a monomorphism, hence $\mathrm{Ext}^i_{{}^p\mathbf{Perv}(X)}(\mathcal{E},\mathcal{F}) \to \mathrm{Ext}^i_{\mathbf{D}_c(X)}(\mathcal{E},\mathcal{F})$ is a monomorphism as well. Finally, since $\mathrm{Ext}^1_{\mathbf{D}_c(X)}(\mathcal{E},\mathcal{F}) \cong \mathrm{Ext}^1_{{}^p\mathbf{Perv}(X)}(\mathcal{E},\mathcal{F})$, see Remark 2.1.5.3, the map $\mathrm{Ext}^2_{{}^p\mathbf{Perv}(X)}(\mathcal{E},\mathcal{F}) \to \mathrm{Ext}^2_{\mathbf{D}_c(X)}(\mathcal{E},\mathcal{F})$ is always a monomorphism. □

Remark 4.2.3.3. *Lemma 4.2.3.2 extends [BGS96, Lemma 3.2.4] where X is assumed to be an algebraic variety and p the middle perversity. Note that Lemma 4.2.3.2 holds without any assumption on the faithfulness of the heart.*

Let us consider the Yoneda product on ${}^p\mathbf{Perv}(X)$, that is

$$\mathrm{m}_2 : \mathrm{Ext}^1(\mathcal{S},\mathcal{S}) \otimes \mathrm{Ext}^1(\mathcal{S},\mathcal{S}) \to \mathrm{Ext}^2_{{}^p\mathbf{Perv}(X)}(\mathcal{S},\mathcal{S}),$$

and the canonical map $\mathrm{Ext}^2_{{}^p\mathbf{Perv}(X)}(\mathcal{S},\mathcal{S}) \to \mathrm{Ext}^2_{\mathbf{D}_c(X)}(\mathcal{S},\mathcal{S})$ which is injective by Lemma

4.2.3.2. The composite

$$\mathrm{m} : \mathrm{Ext}^1(\mathcal{S},\mathcal{S}) \otimes \mathrm{Ext}^1(\mathcal{S},\mathcal{S}) \to \mathrm{Ext}^2_{{}^p\mathbf{Perv}(X)}(\mathcal{S},\mathcal{S}) \hookrightarrow \mathrm{Ext}^2_{\mathbf{D}_c(X)}(\mathcal{S},\mathcal{S}) \tag{4.5}$$

is the composition of morphisms in $\mathbf{D}_c(X)$. Finally, the quadratic part of the relations is the image of the map dual to the one in (4.5), see [LPWZ09, Corollary B] and [Kel01, Proposition 2]. Therefore, we have $\bar{\mathrm{I}}_p(X) = \mathrm{im}(\mathrm{m}^\vee)$ where

$$\mathrm{m}^\vee : \mathrm{Ext}^2_{\mathbf{D}_c(X)}(\mathcal{S},\mathcal{S})^\vee \to \mathrm{Ext}^1(\mathcal{S},\mathcal{S})^\vee \otimes \mathrm{Ext}^1(\mathcal{S},\mathcal{S})^\vee.$$

Our goal is trying to determine the quadratic part of the relations by induction on the number of strata. We will need the following two preliminary results.

Lemma 4.2.3.4. *Let $\mathcal{L},\mathcal{M}$ be local systems on (possibly distinct) strata of X. Then*

$$\mathrm{Ext}^1_{{}^p\mathbf{Perv}(S)}(i^*\mathcal{S}_\mathcal{L}, i^!\mathcal{S}_\mathcal{M}) = 0.$$

Proof. The claim follows since simple objects satisfy the strong vanishing conditions, see Remark 2.3.5.5.iii). Therefore $i^*\mathcal{S}_\mathcal{L} \in {}^p\mathrm{D}^{\leq -1}(S)$ and $i^!\mathcal{S}_\mathcal{M} \in {}^p\mathrm{D}^{\geq 1}(S)$, hence the claim. □

Lemma 4.2.3.5. *Let $\mathcal{L},\mathcal{M}$ be local systems on (possibly distinct) strata of X. Let $S \subset X$ be a closed stratum and $\mathcal{N} \in \mathbf{Loc}(S)$. Then,*

$$\begin{aligned}\mathrm{Ext}^2_{\mathbf{D}_c(S)}(i^*\mathcal{S}_\mathcal{L}, i^!\mathcal{S}_\mathcal{M}) &\cong \bigoplus_{\mathcal{N}\in\mathbf{Loc}(S)} \mathrm{Ext}^1(\mathcal{S}_\mathcal{L},\mathcal{S}_\mathcal{N}) \otimes \mathrm{Ext}^1(\mathcal{S}_\mathcal{N},\mathcal{S}_\mathcal{M})\\ &\cong \bigoplus_{\mathcal{N}\in\mathbf{Loc}(S)} \mathrm{Ext}^1(i^*\mathcal{S}_\mathcal{L},\widehat{\mathcal{S}}_\mathcal{N}) \otimes \mathrm{Ext}^1(\widehat{\mathcal{S}}_\mathcal{N}, i^!\mathcal{S}_\mathcal{M})\end{aligned}$$

where $\widehat{\mathcal{S}}_\mathcal{N} \in {}^p\mathbf{Perv}(S) \simeq \mathbf{Loc}(S)[-p(S)]$ denotes a simple object in the semisimple category of local systems on the stratum S.

Proof. The strong vanishing conditions satisfied by simple objects, see Remark 2.3.5.5.iii), imply

$$i^*\mathcal{S}_\mathcal{L} \in {}^p\mathrm{D}^{\leq -1}(S) \qquad \text{and} \qquad i^!\mathcal{S}_\mathcal{M} \in {}^p\mathrm{D}^{\geq 1}(S).$$

Moreover, as $\mathcal{S}_\mathcal{N} \cong i_*\widehat{\mathcal{S}}_\mathcal{N} \in {}^p\mathrm{D}^0(X)$ we have

$$\mathrm{Ext}^1(\mathcal{S}_\mathcal{L},\mathcal{S}_\mathcal{N}) \cong \mathrm{Ext}^1(i^*\mathcal{S}_\mathcal{L},\widehat{\mathcal{S}}_\mathcal{N}) \cong \mathrm{Hom}({}^pH^{-1}(i^*\mathcal{S}_\mathcal{L}),\widehat{\mathcal{S}}_\mathcal{N})$$

and dually

$$\mathrm{Ext}^1(\mathcal{S}_{\mathcal{N}}, \mathcal{S}_{\mathcal{M}}) \cong \mathrm{Ext}^1(\widehat{\mathcal{S}}_{\mathcal{N}}, i^!\mathcal{S}_{\mathcal{M}}) \cong \mathrm{Hom}(\widehat{\mathcal{S}}_{\mathcal{N}}, {}^pH^1(i^*\mathcal{S}_{\mathcal{M}})).$$

Therefore,

$$\begin{aligned}\mathrm{Ext}^2_{\mathbf{D}_c(S)}(i^*\mathcal{S}_{\mathcal{L}}, i^!\mathcal{S}_{\mathcal{M}}) &\cong \bigoplus_{\mathcal{N}\in\mathbf{Loc}(S)} \mathrm{Hom}({}^pH^{-1}(i^*\mathcal{S}_{\mathcal{L}}), \widehat{\mathcal{S}}_{\mathcal{N}}) \otimes \mathrm{Hom}(\widehat{\mathcal{S}}_{\mathcal{N}}, {}^pH^1(i^!\mathcal{S}_{\mathcal{M}}))\\ &\cong \bigoplus_{\mathcal{N}\in\mathbf{Loc}(S)} \mathrm{Ext}^1(i^*\mathcal{S}_{\mathcal{L}}, \widehat{\mathcal{S}}_{\mathcal{N}}) \otimes \mathrm{Ext}^1(\widehat{\mathcal{S}}_{\mathcal{N}}, i^!\mathcal{S}_{\mathcal{M}})\\ &\cong \bigoplus_{\mathcal{N}\in\mathbf{Loc}(S)} \mathrm{Ext}^1(\mathcal{S}_{\mathcal{L}}, \mathcal{S}_{\mathcal{N}}) \otimes \mathrm{Ext}^1(\mathcal{S}_{\mathcal{N}}, \mathcal{S}_{\mathcal{M}}).\end{aligned}$$

□

We now want to understand how the quadratic part of relations changes when one adds a closed stratum S. We study four different cases depending on the support of the considered irreducible local systems.

Let us consider $\mathcal{L}, \mathcal{M} \notin \mathbf{Loc}(S)$.
In this case, we have the following result:

Proposition 4.2.3.6. *Let $\mathcal{L}, \mathcal{M} \in \mathbf{Loc}(U)$ and $\mathcal{N} \in \mathbf{Loc}(S)$. Then, there is a commutative diagram*

$$\begin{tikzcd}
\mathrm{Ext}^1(j^*\mathcal{S}_{\mathcal{L}}, j^*\mathcal{S}_{\mathcal{M}})^\vee \arrow[r, "\alpha"] & \mathrm{Ext}^1(\mathcal{S}_{\mathcal{L}}, \mathcal{S}_{\mathcal{M}})^\vee \arrow[r] & 0 & \\
\mathrm{Ext}^2_{\mathbf{D}_c(S)}(i^*\mathcal{S}_{\mathcal{L}}, i^!\mathcal{S}_{\mathcal{M}})^\vee \arrow[u] \arrow[d, "\cong"] & \mathrm{Ext}^2_{\mathbf{D}_c(X)}(\mathcal{S}_{\mathcal{L}}, \mathcal{S}_{\mathcal{M}})^\vee \arrow[l, "\gamma"'] \arrow[d, "\mathrm{m}^\vee_{\mathcal{L},\mathcal{M}}"] & \mathrm{Ext}^2_{\mathbf{D}_c(U)}(j^*\mathcal{S}_{\mathcal{L}}, j^*\mathcal{S}_{\mathcal{M}})^\vee \arrow[l] \arrow[d, "\mathrm{m}^\vee_{\mathcal{L},\mathcal{M}}|_U"] & \dots \arrow[l] \\
\bigoplus\limits_{\mathcal{N}\in\mathbf{Loc}(S)} \mathrm{Ext}^1(\mathcal{S}_{\mathcal{L}}, \mathcal{S}_{\mathcal{N}})^\vee \otimes \mathrm{Ext}^1(\mathcal{S}_{\mathcal{N}}, \mathcal{S}_{\mathcal{M}})^\vee & \bigoplus\limits_{\substack{\mathcal{T}\in\mathbf{Loc}(T)\\ T\subset X}} \mathrm{Ext}^1(\mathcal{S}_{\mathcal{L}}, \mathcal{S}_{\mathcal{T}})^\vee \otimes \mathrm{Ext}^1(\mathcal{S}_{\mathcal{T}}, \mathcal{S}_{\mathcal{M}})^\vee \arrow[l, two heads] & \bigoplus\limits_{\substack{\mathcal{T}\in\mathbf{Loc}(T)\\ T\neq S}} \mathrm{Ext}^1(j^*\mathcal{S}_{\mathcal{L}}, j^*\mathcal{S}_{\mathcal{T}})^\vee \otimes \mathrm{Ext}^1(j^*\mathcal{S}_{\mathcal{T}}, j^*\mathcal{S}_{\mathcal{M}})^\vee \arrow[l, hook', "\beta"'] & \dots \arrow[l]
\end{tikzcd} \quad . \tag{4.6}$$

Proof. Let $\mathcal{L}, \mathcal{M} \in \mathbf{Loc}(U)$ and consider the triangle

$$i_*i^!\mathcal{S}_{\mathcal{M}} \to \mathcal{S}_{\mathcal{M}} \to j_*j^*\mathcal{S}_{\mathcal{M}} \to i_*i^!\mathcal{S}_{\mathcal{M}}[1]$$

in $\mathbf{D}_{\mathrm{c}}(X)$. By applying the functor $\mathrm{Ext}^i_{\mathbf{D}_{\mathrm{c}}(X)}(\mathcal{S}_{\mathcal{L}}, -)$ we get the long exact sequence

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Ext}^1(\mathcal{S}_{\mathcal{L}}, i_*i^!\mathcal{S}_{\mathcal{M}}) \arrow[r] & \mathrm{Ext}^1(\mathcal{S}_{\mathcal{L}}, \mathcal{S}_{\mathcal{M}}) \arrow[r] & \mathrm{Ext}^1(\mathcal{S}_{\mathcal{L}}, j_*j^*\mathcal{S}_{\mathcal{M}}) \arrow[dll, out=0, in=180] & \\
& \mathrm{Ext}^2_{\mathbf{D}_{\mathrm{c}}(X)}(\mathcal{S}_{\mathcal{L}}, i_*i^!\mathcal{S}_{\mathcal{M}}) \arrow[r] & \mathrm{Ext}^2_{\mathbf{D}_{\mathrm{c}}(X)}(\mathcal{S}_{\mathcal{L}}, \mathcal{S}_{\mathcal{M}}) \arrow[r] & \mathrm{Ext}^2_{\mathbf{D}_{\mathrm{c}}(X)}(\mathcal{S}_{\mathcal{L}}, j_*j^*\mathcal{S}_{\mathcal{M}}) \arrow[r] & \dots
\end{tikzcd}$$

which, by adjunctions and Lemma 4.2.3.4, becomes

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Ext}^1(\mathcal{S}_{\mathcal{L}}, \mathcal{S}_{\mathcal{M}}) \arrow[r] & \mathrm{Ext}^1(j^*\mathcal{S}_{\mathcal{L}}, j^*\mathcal{S}_{\mathcal{M}}) \arrow[dll, out=0, in=180] & \\
\mathrm{Ext}^2_{\mathbf{D}_{\mathrm{c}}(S)}(i^*\mathcal{S}_{\mathcal{L}}, i^!\mathcal{S}_{\mathcal{M}}) \arrow[r] & \mathrm{Ext}^2_{\mathbf{D}_{\mathrm{c}}(X)}(\mathcal{S}_{\mathcal{L}}, \mathcal{S}_{\mathcal{M}}) \arrow[r] & \mathrm{Ext}^2_{\mathbf{D}_{\mathrm{c}}(U)}(j^*\mathcal{S}_{\mathcal{L}}, j^*\mathcal{S}_{\mathcal{M}}) \arrow[r] & \dots
\end{tikzcd}$$

Moreover, for a local system $\mathcal{N} \in \mathbf{Loc}(S)$ one can build the following commutative diagram

$$\begin{tikzcd}
\mathrm{Ext}^1(j^*\mathcal{S}_{\mathcal{L}}, j^*\mathcal{S}_{\mathcal{M}}) \arrow[d] & \mathrm{Ext}^1(\mathcal{S}_{\mathcal{L}}, \mathcal{S}_{\mathcal{M}}) \arrow[l] & 0 \arrow[l] & \\
\mathrm{Ext}^2_{\mathbf{D}_{\mathrm{c}}(S)}(i^*\mathcal{S}_{\mathcal{L}}, i^!\mathcal{S}_{\mathcal{M}}) \arrow[r] & \mathrm{Ext}^2_{\mathbf{D}_{\mathrm{c}}(X)}(\mathcal{S}_{\mathcal{L}}, \mathcal{S}_{\mathcal{M}}) \arrow[r] & \mathrm{Ext}^2_{\mathbf{D}_{\mathrm{c}}(U)}(j^*\mathcal{S}_{\mathcal{L}}, j^*\mathcal{S}_{\mathcal{M}}) \arrow[r] & \dots \\
\bigoplus\limits_{\mathcal{N} \in \mathbf{Loc}(S)} \mathrm{Ext}^1(\mathcal{S}_{\mathcal{L}}, \mathcal{S}_{\mathcal{N}}) \otimes \mathrm{Ext}^1(\mathcal{S}_{\mathcal{N}}, \mathcal{S}_{\mathcal{M}}) \arrow[u, "\delta"] \arrow[r, hook] & \bigoplus\limits_{\substack{\mathcal{T} \in \mathbf{Loc}(T) \\ T \subset X}} \mathrm{Ext}^1(\mathcal{S}_{\mathcal{L}}, \mathcal{S}_{\mathcal{T}}) \otimes \mathrm{Ext}^1(\mathcal{S}_{\mathcal{T}}, \mathcal{S}_{\mathcal{M}}) \arrow[u] \arrow[r] & \bigoplus\limits_{\substack{\mathcal{T} \in \mathbf{Loc}(T) \\ T \neq S}} \mathrm{Ext}^1(j^*\mathcal{S}_{\mathcal{L}}, j^*\mathcal{S}_{\mathcal{T}}) \otimes \mathrm{Ext}^1(j^*\mathcal{S}_{\mathcal{T}}, j^*\mathcal{S}_{\mathcal{M}}) \arrow[u] \arrow[r] & \dots
\end{tikzcd} \tag{4.7}$$

where the squares commute and δ is an isomorphism by Lemma 4.2.3.5. Therefore, dualising (4.7), one gets the claim. $\square$

We now want to interpret the diagram (4.6) in a more algebraic way. We will denote the idempotents of the basic algebra $\mathrm{A}_p(X)$ by $e_{\mathcal{L}}$. Note that $e_{\mathcal{L}}$ corresponds to the projective cover of the simple object $\mathcal{S}_{\mathcal{L}} \in {}^p\mathbf{Perv}(X)$, where $\mathcal{L} \in \mathbf{Loc}(X)$ is an irreducible local system supported on some strata of X. Therefore, we will use the notation $e_{\mathcal{L}}\mathrm{A}_p(X)e_{\mathcal{M}}$ to denote the paths $\mathcal{L} \to \mathcal{M}$ in $\mathrm{A}_p(X)$, that is by [ASS06, III Lemma 2.12]

$$e_{\mathcal{L}}\mathrm{A}_p(X)e_{\mathcal{M}} \cong \mathrm{Hom}_{{}^p\mathbf{Perv}(X)}(\mathcal{P}_{\mathcal{M}}, \mathcal{P}_{\mathcal{L}}). \tag{4.8}$$

Using this notation, we have that (4.6) becomes the following exact sequence

$$\begin{tikzcd}
0 & e_{\mathcal{L}}\mathrm{A}^1_p(X)e_{\mathcal{M}} \arrow[l] & e_{\mathcal{L}}\mathrm{A}^1_p(U)e_{\mathcal{M}} \arrow[l] & \bigoplus_{\mathcal{N}\in\mathbf{Loc}(S)} e_{\mathcal{L}}\mathrm{A}^1_p(X)e_{\mathcal{N}}\mathrm{A}^1_p(X)e_{\mathcal{M}} \arrow[l] \\
0 \arrow[r] & e_{\mathcal{L}}(\bar{\mathrm{I}}_p(U)\cap \mathrm{J}_p(U))e_{\mathcal{M}} \arrow[r] & e_{\mathcal{L}}\bar{\mathrm{I}}_p(U)e_{\mathcal{M}} \arrow[r] & e_{\mathcal{L}}\bar{\mathrm{I}}_p(X)e_{\mathcal{M}} \arrow[u]
\end{tikzcd} \tag{4.9}$$

where $\mathrm{J}_p(U)$ is the ideal generated by vanishing arrows coming from U, that is

$$\mathrm{J}_p(U) = \ker(e_{\mathcal{E}}\mathrm{A}^1_p(U)e_{\mathcal{F}} \twoheadrightarrow e_{\mathcal{E}}\mathrm{A}^1_p(X)e_{\mathcal{F}}) \quad \text{for any} \quad \mathcal{E},\mathcal{F} \in \bigcup_{T\subset U} \mathbf{Loc}(T).$$

The exact sequence (4.9) governs how arrows and relations between vertices $\mathcal{L}$ and $\mathcal{M}$ coming from U change. In this case, the simplest situation is when

$$\bigoplus_{\mathcal{N}\in\mathbf{Loc}(S)} e_{\mathcal{L}}\mathrm{A}^1_p(X)e_{\mathcal{N}}\mathrm{A}^1_p(X)e_{\mathcal{M}} \cong 0,$$

that is when there are no new composites via new vertices $\mathcal{N} \in \mathbf{Loc}(S)$. This implies that (4.9) splits in the isomorphism

$$e_{\mathcal{L}}\mathrm{A}^1_p(X)e_{\mathcal{M}} \cong e_{\mathcal{L}}\mathrm{A}^1_p(U)e_{\mathcal{M}}$$

and in the short exact sequence

$$0 \to e_{\mathcal{L}}(\bar{\mathrm{I}}_p(U)\cap \mathrm{J}_p(U))e_{\mathcal{M}} \to e_{\mathcal{L}}\bar{\mathrm{I}}_p(U)e_{\mathcal{M}} \to e_{\mathcal{L}}\bar{\mathrm{I}}_p(X)e_{\mathcal{M}} \to 0,$$

that is the arrows in $\mathrm{Q}_p(X)$ between vertices corresponding to local systems on strata in U coincides with the ones in $\mathrm{Q}_p(U)$ and there are no new relations. Therefore, in order to obtain relations in $\bar{\mathrm{I}}_p(X)$ one needs to take those in $\bar{\mathrm{I}}_p(U)$ and delete terms in $\mathrm{J}_p(U)$.

More generally, it is always true that taking a relation in $e_{\mathcal{L}}\bar{\mathrm{I}}_p(U)e_{\mathcal{M}}$ and deleting terms in $\mathrm{J}_p(U)$ gives rise to a relation in $e_{\mathcal{L}}\bar{\mathrm{I}}_p(X)e_{\mathcal{M}}$ and that all relations in $e_{\mathcal{L}}\bar{\mathrm{I}}_p(X)e_{\mathcal{M}}$ which do not involve paths via a vertex $\mathcal{N} \in \mathbf{Loc}(S)$ arise in this way. Moreover, for any path in

$$\ker(e_{\mathcal{L}}\mathrm{A}^1_p(U)e_{\mathcal{M}} \to e_{\mathcal{L}}\mathrm{A}^1_p(X)e_{\mathcal{M}})$$

one can choose a two step path in

$$\bigoplus_{\mathcal{N} \in \mathbf{Loc}(S)} e_{\mathcal{L}} \mathrm{A}^1_p(X) e_{\mathcal{N}} \mathrm{A}^1_p(X) e_{\mathcal{M}},$$

that is via vertices in $\mathbf{Loc}(S)$, uniquely up to $e_{\mathcal{L}}\bar{\mathrm{I}}_p(X)e_{\mathcal{M}}$, as in the following schematic picture

$$\begin{tikzcd}
\mathcal{L} \arrow[dd, "\alpha"] & & \mathcal{L} \arrow[dr, "\beta"] & \\
& & & \mathcal{N} \arrow[dl, "\gamma"] \\
\mathcal{M} & & \mathcal{M} &
\end{tikzcd} .$$

Remark 4.2.3.7. *Note that, using that ${}^pj_!$ is fully faithful and (4.8), the above situation is consistent with the fact that*

$$\begin{aligned}
e_{\mathcal{L}}\mathrm{A}_p(X)e_{\mathcal{M}} &\cong \mathrm{Hom}_{{}^p\mathbf{Perv}(X)}(\mathcal{P}_{\mathcal{L}}, \mathcal{P}_{\mathcal{M}}) \cong \mathrm{Hom}_{{}^p\mathbf{Perv}(X)}({}^pj_!\widehat{\mathcal{P}}_{\mathcal{L}}, {}^pj_!\widehat{\mathcal{P}}_{\mathcal{M}}) \\
&\cong \mathrm{Hom}_{{}^p\mathbf{Perv}(U)}(\widehat{\mathcal{P}}_{\mathcal{L}}, \widehat{\mathcal{P}}_{\mathcal{M}}) \cong e_{\mathcal{L}}\mathrm{A}_p(U)e_{\mathcal{M}},
\end{aligned}$$

where $\widehat{\mathcal{P}}_{\mathcal{L}} \in {}^p\mathbf{Perv}(U)$ denotes the projective cover of the simple object $\mathcal{S}_{\mathcal{L}}$ in ${}^p\mathbf{Perv}(U)$.

More geometrically, we can phrase Remark 4.2.3.7 as follows.

Lemma 4.2.3.8. *Let $U \subset X$ be an open stratum and $\mathcal{L}, \mathcal{M} \in \mathbf{Loc}(U)$. Then, any non-trivial path in $\mathrm{Q}_p(X)$ from $\mathcal{L}$ to $\mathcal{M}$ is in the ideal $\mathrm{I}_p(X)$.*

Proof. Let $j : U \hookrightarrow X$ be the open inclusion. Then the projective cover of $\mathcal{S}_{\mathcal{L}}$ in ${}^p\mathbf{Perv}(X)$ is $\mathcal{P}_{\mathcal{L}} \cong {}^pj_!\mathcal{L}[-p(S)]$, see Proposition 3.3.1.1. Therefore, if we consider the space of paths between $\mathcal{M}$ and $\mathcal{L}$ in $\Bbbk\mathrm{Q}_p(X)/\mathrm{I}_p(X)$, we have

$$\begin{aligned}
\mathrm{Hom}_{{}^p\mathbf{Perv}(X)}(\mathcal{P}_{\mathcal{L}}, \mathcal{P}_{\mathcal{M}}) &\cong \mathrm{Hom}_{{}^p\mathbf{Perv}(X)}({}^pj_!\mathcal{L}[-p(U)], {}^pj_!\mathcal{M}[-p(U)]) \\
&\cong \mathrm{Hom}_{\mathbf{Loc}(U)}(\mathcal{L}, \mathcal{M}) \\
&\cong \begin{cases} \Bbbk & \text{if } \mathcal{L} = \mathcal{M} \\ 0 & \text{otherwise} \end{cases}.
\end{aligned}$$

□

Example 4.2.3.9. *Let us consider the situation of Example 4.1.2.1.i). For any* $\mathcal{L}, \mathcal{M} \in \mathbf{Loc}(S)$ *where* $S \subset U$*, Lemma 4.2.3.8 implies that any non-trivial path from* $\mathcal{L}$ *to* $\mathcal{M}$ *in* $\mathrm{Q}_m(\mathbb{P}^1)$ *is a relation. Therefore* $\beta \circ \alpha = 0$ *in (4.3), that is* $\beta \circ \alpha$ *is a relation or equivalently the cycle around the vertex* $\Bbbk_U$ *in* $\mathrm{Q}_m(\mathbb{P}^1)$ *is zero.*

Returning to the general case, paths in

$$\ker\left(\bigoplus_{\mathcal{N}\in\mathbf{Loc}(S)} e_{\mathcal{L}}\mathrm{A}^1_p(X)e_{\mathcal{N}}\mathrm{A}^1_p(X)e_{\mathcal{M}} \to e_{\mathcal{L}}\mathrm{A}^1_p(U)e_{\mathcal{M}}\right)$$

must appear in a new relation, which might have other quadratic terms as well.

Let us consider $\mathcal{L} \in \mathbf{Loc}(S)$ and $\mathcal{M} \notin \mathbf{Loc}(S)$ (or dually $\mathcal{M} \in \mathbf{Loc}(S)$ and $\mathcal{L} \notin \mathbf{Loc}(S)$).

In this situation, there is nothing to point out except for the fact that $e_{\mathcal{L}}\bar{\mathrm{I}}_p(X)e_{\mathcal{M}} = 0$ if $e_{\mathcal{L}}\mathrm{A}^1_p(X)e_{\mathcal{N}} = 0$ for any $\mathcal{N}$ or $e_{\mathcal{N}}\mathrm{A}^1_p(X)e_{\mathcal{M}} = 0$ for any $\mathcal{N}$ respectively. In this case, the best strategy to find the quadratic part of relations $\bar{\mathrm{I}}_p(X)$ is to compute the Ext-algebra.

Let us consider $\mathcal{L}, \mathcal{M} \in \mathbf{Loc}(S)$.

In this case, the only thing to note is that there is a surjection $H^2(S; \mathcal{L}^\vee \otimes \mathcal{M}) \twoheadrightarrow e_{\mathcal{L}}\bar{\mathrm{I}}_p(X)e_{\mathcal{M}}$. Therefore, we have the following result.

Lemma 4.2.3.10. *Let* $S \subset X$ *be a closed stratum and consider* $\mathcal{L}, \mathcal{M} \in \mathbf{Loc}(S)$*. If* $H^2(S; \mathcal{L}^\vee \otimes \mathcal{M}) = 0$ *for any* $\mathcal{L} \in \mathbf{Loc}(S)$*, there are no relations between paths in* Q_p *from* $\mathcal{L}$ *to* $\mathcal{M}$*.*

Proof. This follows from the fact that $\mathrm{Ext}^2_{\mathbf{D}_c(X)}(\mathcal{S}_{\mathcal{L}}, \mathcal{S}_{\mathcal{M}}) \cong H^2(S; \mathcal{L}^\vee \otimes \mathcal{M}) = 0$. □

Example 4.2.3.11. *Let us consider the situation of Example 4.1.2.1.i). For any* $\mathcal{L} \in \mathbf{Loc}(Z)$ *where* $Z \cong \{\mathrm{pt}\}$*, we have* $H^2(Z; \mathcal{L}) \cong 0$*. Hence Lemma 4.2.3.10 implies that for any other* $\mathcal{M} \in \mathbf{Loc}(Z)$ *there are no relations between paths from* $\mathcal{L}$ *to* $\mathcal{M}$ *in* $\mathrm{Q}_m(X)$*. In particular, we have* $\alpha \circ \beta \neq 0$ *in (4.3), that is* $\alpha \circ \beta$ *is not a relation.*

Remark 4.2.3.12. *Note that Proposition 4.2.2.5 implies that Koszul algebras are quadratic. Therefore, for that class of algebras the above procedure allows us at least in principle to determine the ideal of relations* $\mathrm{I}_p(X) = \bar{\mathrm{I}}_p(X)$*. In some other examples, namely for affine* *stratifications of complex varieties where the closure of strata admit more manageable resolutions, the* A_∞*-structure is formal hence all the relations are quadratic, see [BGS96].*

Remark 4.2.3.13. *Note that Examples 4.1.2.1, 4.2.3.11 and 4.2.3.9 give the* Ext*-quiver with relations for the category* ${}^{m}\mathbf{Perv}(X)$ *in purely topological terms, without using for example the machinery of nearby and vanishing cycles.*

4.3 Topology of Links

Diagram (4.6) shows that in order to determine inductively how the arrows and the quadratic part of the ideal of relations change when one adds a closed stratum it is enough to control the groups $\mathrm{Ext}^1(\mathcal{S}_\mathcal{L}, \mathcal{S}_\mathcal{N})$ for $\mathcal{L} \in \mathbf{Loc}(T)$ for some stratum $T \subset U$ and $\mathcal{N} \in \mathbf{Loc}(S)$ (and dually $\mathrm{Ext}^1(\mathcal{S}_\mathcal{N}, \mathcal{S}_\mathcal{M})$ for $\mathcal{M} \in \mathbf{Loc}(U)$). In this section, we give an explicit characterisation of Ext^k-groups in terms of intersection cohomology groups. We then relate the cases $k = 1$ and $k = 2$, that is the cases that give arrows and relations respectively, to some specific intersection cohomology groups of links.

Let $\mathcal{L} \in \mathbf{Loc}(T)$ for some stratum $T \subset U$ and $\mathcal{N} \in \mathbf{Loc}(S)$. Let us denote by $i : S \hookrightarrow X$ the inclusion of a closed stratum into X and by $\widehat{\mathcal{S}}_\mathcal{N} \in {}^{p}\mathbf{Perv}(S) \simeq \mathbf{Loc}(S)[-p(S)]$ the simple perverse sheaves supported on S. We will make the following assumptions:

a) $\pi_1(S) = 0$ for any stratum $S \subset X$.

b) p is a GM-perversity.

We then have:

$$
\begin{aligned}
\mathrm{Ext}^k_{\mathbf{D}_\mathrm{c}(X)}(\mathcal{S}_\mathcal{L}, \mathcal{S}_\mathcal{N}) &\cong \mathrm{Ext}^k_{\mathbf{D}_\mathrm{c}(X)}(\mathcal{S}_\mathcal{L}, i_*\widehat{\mathcal{S}}_\mathcal{N}) \cong \mathrm{Ext}^k_{\mathbf{D}_\mathrm{c}(S)}(i^*\mathcal{S}_\mathcal{L}, \widehat{\mathcal{S}}_\mathcal{N}) \\
&\cong \mathrm{Ext}^k_{\mathbf{D}_\mathrm{c}(S)}(i^*\mathcal{S}_\mathcal{L}, \mathcal{N}[-p(S)]) \cong \mathrm{Hom}_{\mathbf{D}_\mathrm{c}(S)}(i^*\mathcal{S}_\mathcal{L}[p(S)-k], \mathcal{N}) \\
&\cong \mathrm{Hom}_{\mathbf{Loc}(S)}(H^0(i^*\mathcal{S}_\mathcal{L})[p(S)-k], \mathcal{N}) \quad \text{as } \mathcal{N} \in \mathrm{D}^0(S) \ \text{ and } \mathbf{D}_\mathrm{c}(S) \text{ semisimple}, \\
&\cong \mathrm{Hom}_{\mathbf{Loc}(S)}(H^{p(S)-k}(i^*\mathcal{S}_\mathcal{L}), \mathcal{N}) \\
&\cong H^{p(S)-k}_x(i^*\mathcal{S}_\mathcal{L})^\vee \quad \text{as } \pi_1(S) = 0 \ \Rightarrow\ \mathcal{N} \cong \Bbbk_S, \\
&\cong {}^{p}H^{p(S)-p(T)-k}(L_S; \mathcal{L}|_{L_S})^\vee
\end{aligned}
\tag{4.10}
$$

where $x \in S \subset X$ and L_S is the link of S. Dually to (4.10), for $\mathcal{M} \in \mathbf{Loc}(T)$ for some stratum $T \subset U$, we have

$$\operatorname{Ext}^k_{\mathbf{D}_c(X)}(\mathcal{S}_{\mathcal{N}}, \mathcal{S}_{\mathcal{M}}) \cong H_x^{p(S)+k}(i^! \mathcal{S}_{\mathcal{M}}) \cong {}^pH^{p(S)+p(T)+k}(L_S; \mathcal{M}|_{L_S}). \tag{4.11}$$

Therefore, in order to be able to run the argument in diagram (4.6), one needs to calculate:

i) The groups

$$\operatorname{Ext}^1(\mathcal{S}_{\mathcal{L}}, \mathcal{S}_{\mathcal{N}}) \cong H_x^{p(S)-1}(i^* \mathcal{S}_{\mathcal{L}})^\vee \cong {}^pH^{p(S)-p(T)-1}(L_S; \mathcal{L}_{L_S})^\vee$$

and

$$\operatorname{Ext}^1(\mathcal{S}_{\mathcal{N}}, \mathcal{S}_{\mathcal{M}}) \cong H_x^{p(S)+1}(i^! \mathcal{S}_{\mathcal{L}}) \cong {}^pH^{p(S)+p(T)+1}(L_S; \mathcal{M}|_{L_S})$$

for any $\mathcal{L}, \mathcal{M} \in \mathbf{Loc}(T)$ for some $T \subset U$. Indeed, these groups control how new arrows of the form

$$\mathcal{L} \to \mathcal{N} \qquad \text{and} \qquad \mathcal{N} \to \mathcal{M}$$

appear in the Ext-quiver $\mathrm{Q}_p(X)$ when a closed stratum $S \subset X$ is added.

ii) The groups

$$\operatorname{Ext}^2_{\mathbf{D}_c(X)}(\mathcal{S}_{\mathcal{L}}, \mathcal{S}_{\mathcal{N}}) \cong {}^pH^{p(S)-p(T)-2}(L_S; \mathcal{L}|_{L_S})^\vee$$

and

$$\operatorname{Ext}^2_{\mathbf{D}_c(X)}(\mathcal{S}_{\mathcal{N}}, \mathcal{S}_{\mathcal{M}}) \cong {}^pH^{p(S)+p(T)+2}(L_S; \mathcal{M}|_{L_S})$$

for any $\mathcal{L}, \mathcal{M} \in \mathbf{Loc}(T)$ for some $T \subset U$. Indeed, these groups control how new relations on paths from $\mathcal{L}$ to $\mathcal{N}$ and $\mathcal{N}$ to $\mathcal{M}$ appears in the Ext-quiver $\mathrm{Q}_p(X)$ when one adds a closed stratum $S \subset X$.

Note that, if we replace the assumption a) with $\pi_1(S)$ finite for any stratum $S \subset X$, one obtains the twisted version of (4.10) and (4.11). On the other hand, if condition b) fails, it is not very clear what the definition of intersection cohomology is, therefore this calculation does not help.

4.4 Perverse Sheaves as Quiver Representations

Let X be a topologically stratified space with finitely many strata S, each with finite fundamental group. Let p be a perversity on X and $\Bbbk$ an algebraically closed field such that its characteristic does not divide the order of $\pi_1(S)$ for any stratum $S \subset X$. In Section 3.4, we gave different characterisations of the category of p-perverse sheaves. In particular, Theorem 2.2.2.14 gives the equivalence

$$^p\mathbf{Perv}(X) \simeq \mathbf{rep}(\mathrm{Q}_p(X), \mathrm{I}_p(X)),$$

where $\mathrm{Q}_p(X)$ is the Ext-quiver of the category $^p\mathbf{Perv}(X)$ and $\mathrm{I}_p(X)$ the ideal of relations. This approach is particularly helpful because it reduces some problems to linear algebra questions. In the notation of Definition 2.2.2.1, let us denote the Ext-quiver of the category $^p\mathbf{Perv}(X)$ by $\mathrm{Q}_p(X) = (\mathrm{Q}_0, \mathrm{Q}_1)$.

Simple Representations

For $a \in \mathrm{Q}_0$, define the representation $S(a) = (S(a)_b, \phi_a) \in \mathbf{rep}(\mathrm{Q}_p(X), \mathrm{I}_p(X))$ as

$$S(a)_b = \begin{cases} \Bbbk & \text{if } b = a \\ 0 & \text{otherwise} \end{cases}$$
$$\phi_a = 0 \ \ \forall \alpha \in \mathrm{Q}_1.$$

The set $\{S(a) \mid a \in \mathrm{Q}_0\}$ is a complete set of simple representations in $\mathbf{rep}(\mathrm{Q}_p(X), \mathrm{I}_p(X))$, hence it is a complete set of representative of isomorphism classes of simple $\mathrm{A}_p(X)$-modules where $\mathrm{A}_p(X) \cong \Bbbk\mathrm{Q}_p(X)/\mathrm{I}_p(X)$, see [ASS06, III.2.1] and of simple p-perverse sheaves as well.

Example 4.4.0.1. *Let $X = \mathbb{P}^1$ stratified as in Example 2.3.1.6.iii) with the middle perversity on it, that is $p = m$. As calculated in Example 4.1.2.1.i) the* Ext*-quiver for $^m\mathbf{Perv}(X)$ is given by*

$$\mathrm{Q}_m(X) = \begin{tikzcd} 1 \arrow[r, bend left, "\alpha"] & 0 \arrow[l, bend left, "\beta"] \end{tikzcd} .$$

The two simple representations corresponding to the two simple objects in $\mathcal{S}_U$ and $\mathcal{S}_Z$ in $^m\mathbf{Perv}(X)$ are

$$S(1) = \begin{tikzcd} \Bbbk \arrow[r, bend left] & 0 \arrow[l, bend left] \end{tikzcd}$$

and

$$S(0) = \begin{tikzcd} 0 \arrow[r, bend left] & \Bbbk \arrow[l, bend left] \end{tikzcd}$$

respectively.

Remark 4.4.0.2. *For instance, as noted above, the isomorphism*

$$\mathrm{Hom}_{^p\mathbf{Perv}(X)}(\mathcal{S}_{\mathcal{L}}, \mathcal{S}_{\mathcal{M}}) \cong \begin{cases} \Bbbk & \text{if } \mathcal{L} = \mathcal{M} \\ 0 & \text{otherwise} \end{cases}$$

translates in terms of representations of simple objects to the fact that the only non-zero morphism of representations $f \in \mathrm{Hom}_{\mathbf{rep}(\mathrm{Q}_p(X),\mathrm{I}_p(X))}(S(a), S(b))$ *is the identity* $\mathrm{id}_a : S(a) \to S(a)$.

Indecomposable Projective Representations

The algebra $\mathrm{A}_p(X)$ is basic, see Definition 2.2.1.4, with a complete set of primitive orthogonal idempotents given by $\{e_a \mid a \in \mathrm{Q}_0\}$. Therefore, we have a decomposition

$$\mathrm{A}_p(X) \cong \bigoplus_{a \in \mathrm{Q}_0} e_a \mathrm{A}_p(X)$$

into a direct sum of indecomposable projective $\mathrm{A}_p(X)$-modules. In particular, see [ASS06, III Lemma 2.4], we can define the indecomposable projective representations

$$P(a) = (P(a)_b, \phi_\beta) \in \mathbf{rep}(\mathrm{Q}_p(X), \mathrm{I}_p(X))$$

where

> $P(a)_b$ is the $\Bbbk$-vector space with basis given by the set of $\overline{w} = w + \mathrm{I}_p$ where w is a path from a to b.
>
> $\phi_\beta : P(a)_b \to P(a)_c$ is a $\Bbbk$-linear map associated to any arrow $\beta : b \to c$ given by right multiplication by $\overline{\beta} = \beta + \mathrm{I}_p$.

Remark 4.4.0.3. *If the ideal of relations of the* Ext*-quiver* $\mathrm{Q}_p(X)$ *is zero, that is* $\mathrm{I}_p(X) = 0$*, in order to find the projective cover of an object from its quiver representation it is enough to propagate the non-zero vertices along all paths starting at them.*

Example 4.4.0.4. *Let us consider* $X = \mathbb{R}^2$ *stratified as in Example 2.3.1.6. As calculated* *in Example 4.1.2.5, the* Ext*-quiver for the zero perversity is*

$$\mathrm{Q}_o = \ \Bbbk_{S_2} \longleftarrow \Bbbk_{S_1} \longleftarrow \Bbbk_{S_0}$$

and the ideal of relations is zero, see Example 4.2.2.7. The simple representations are

$$S(2) = \ \Bbbk \longleftarrow 0 \longleftarrow 0 \ ,$$

$$S(1) = \ 0 \longleftarrow \Bbbk \longleftarrow 0 \ ,$$

$$S(0) = \ 0 \longleftarrow 0 \longleftarrow \Bbbk \ .$$

Applying the observation of Remark 4.4.0.3, we have that the projective covers of the three simple objects are

$$P(2) = \ \Bbbk \longleftarrow 0 \longleftarrow 0 \ ,$$

$$P(1) = \ \Bbbk \longleftarrow \Bbbk \longleftarrow 0 \ ,$$

$$P(0) = \ \Bbbk \longleftarrow \Bbbk \longleftarrow \Bbbk \ .$$

Note that $P(2) \cong S(2)$*, therefore the object* $\mathcal{S}_2 \in {}^{o}\mathbf{Perv}(X) \simeq \mathbf{Constr}(X)$ *is simple and projective.*

4.5 Quiver and Relations from Indecomposable Projectives

In this section we explain a general procedure which allows us to determine the quiver $\mathrm{Q}_p(X)$ and extract the ideal of relation $\mathrm{I}_p(X)$ of the category ${}^{p}\mathbf{Perv}(X)$. We will use the construction of projective covers presented in Section 3.3. Let X be a topologically stratified space with finitely many strata, each with finite fundamental group and $\Bbbk$ a field such that its characteristic does not divide the order of $\pi_1(S)$ for any stratum $S \subset X$. Note that, although this approach is very general, in practice it requires a lot of information on projective covers and maps between them, as pointed out at the end of the section.

4.5.1 Projective Quiver

The Ext-quiver $\mathrm{Q}_p(X)$ can be equivalently defined in terms of indecomposable projective objects and (a class of) maps between them. Indeed, we can define the Proj-quiver $\mathrm{P}_p(X)$ of ${}^{p}\mathbf{Perv}(X)$ by saying that vertices in $\mathrm{P}_p(X)$ correspond to projective covers $\mathcal{P}_{\mathcal{L}}$ of simple

p-perverse sheaves $\mathcal{S}_{\mathcal{L}} \in {}^p\mathbf{Perv}(X)$. Recall that, for a bounded quiver algebra $\mathrm{A}_p(X) \cong \Bbbk\mathrm{Q}_p(X)/\mathrm{I}_p(X)$, $\mathrm{rad}(\mathrm{A}_p(X))$ is the radical ideal and $\mathrm{rad}^2(\mathrm{A}_p(X))$ is the ideal generated by paths of length two in $\mathrm{A}_p(X)$. Moreover, $\mathrm{Hom}_{^p\mathbf{Perv}(X)}(\mathcal{P}_{\mathcal{M}}, \mathcal{P}_{\mathcal{L}})$ has a basis given by the set of all paths from $\mathcal{L}$ to $\mathcal{M}$ modulo relations, see (4.8). The number of arrows between $\mathcal{L}$ and $\mathcal{M}$ is given by the dimension of the vector space

$$\begin{aligned}\mathrm{Hom}_{^p\mathbf{Perv}(X)}(\mathcal{P}_{\mathcal{M}}, \mathrm{rad}(\mathcal{P}_{\mathcal{L}})/\mathrm{rad}^2(\mathcal{P}_{\mathcal{L}})) &\cong \mathrm{Hom}_{^p\mathbf{Perv}(X)}(\mathcal{P}_{\mathcal{M}}, \mathrm{rad}(\mathcal{P}_{\mathcal{L}}))/\mathrm{Hom}_{^p\mathbf{Perv}(X)}(\mathcal{P}_{\mathcal{M}}, \mathrm{rad}^2(\mathcal{P}_{\mathcal{L}}))\\ &\cong \underline{\mathrm{Hom}}_{^p\mathbf{Perv}(X)}(\mathcal{P}_{\mathcal{M}}, \mathcal{P}_{\mathcal{L}}),\end{aligned}$$

that is by the dimension of the vector space of morphisms from the object $\mathcal{P}_{\mathcal{M}}$ to $\mathcal{P}_{\mathcal{L}}$ that do not factor through another projective object, see Definition 2.2.3.16. Relations in $\mathrm{P}_p(X)$ are given by (composition of) morphisms which are zero or factor through another indecomposable projective object. Note that the isomorphism

$$\mathrm{Ext}^1_{^p\mathbf{Perv}(X)}(\mathcal{S}_{\mathcal{L}}, \mathcal{S}_{\mathcal{M}})^\vee \cong \mathrm{Hom}_{^p\mathbf{Perv}(X)}(\mathcal{P}_{\mathcal{M}}, \mathrm{rad}(\mathcal{P}_{\mathcal{L}})/\mathrm{rad}^2(\mathcal{P}_{\mathcal{L}}))$$

see [Ben98, Proof of Proposition 2.4.3] and [ASS06, III.2.12] together with the one-to-one correspondence between simple objects and their projective covers gives the bijection

$$\mathrm{Q}_p(X) \longleftrightarrow \mathrm{P}_p(X).$$

Example 4.5.1.1. *Let us consider $X = \mathbb{P}^1$ stratified as in Example 2.3.1.6.iii) with the middle perversity on it, that is $p = m$. The* Proj-*quiver* $\mathrm{P}_m(\mathbb{P}^1)$ *has two vertices corresponding to the projective covers $\mathcal{P}_U$ and $\mathcal{P}_Z$ of the two simple objects $\mathcal{S}_U$ and $\mathcal{S}_Z$ respectively; we label the two vertices by 1 and 0 respectively. There are arrows $\gamma : 1 \to 0$ and $\delta : 0 \to 1$ corresponding to the morphisms*

$$\gamma : \mathcal{P}_Z \to \mathcal{P}_U \qquad \textit{and} \qquad \delta : \mathcal{P}_U \to \mathcal{P}_Z$$

respectively. Moreover, the composite

$$\delta \circ \gamma : \mathcal{P}_Z \to \mathcal{P}_U \to \mathcal{P}_Z$$

is not zero while

$$\gamma \circ \delta : \mathcal{P}_U \to \mathcal{P}_Z \to \mathcal{P}_U$$

is zero. Therefore, we can then draw the Proj-*quiver* $\mathrm{P}_m(\mathbb{P}^1)$ *as*

$$\mathrm{P}_m(\mathbb{P}^1) = \begin{tikzcd} \mathcal{P}_U \arrow[r, bend left, "\gamma"] & \mathcal{P}_Z \arrow[l, bend left, "\delta"] \end{tikzcd}, \tag{4.12}$$

with relation $\delta \circ \gamma = 0$ *(that is the cycle at* $\mathcal{P}_U$ *is zero). Note that one can easily check the claims about these morphisms using the quiver representations for irreducible projective objects.*

Note that the approach via the projective quiver, although equivalent to the Ext-quiver one, in practice requires to be able to explicitly compute:

i) $\mathcal{P}_{\mathcal{L}}$ for any irreducible local system $\mathcal{L} \in \mathbf{Loc}(X)$.

ii) $\mathrm{Hom}(\mathcal{P}_{\mathcal{L}}, \mathcal{P}_{\mathcal{M}})$ for every $\mathcal{L}, \mathcal{M} \in \mathbf{Loc}(X)$ irreducible local systems.

iii) $\mathrm{Hom}(\mathcal{P}_{\mathcal{L}}, \mathcal{P}_{\mathcal{M}}) \otimes \mathrm{Hom}(\mathcal{P}_{\mathcal{M}}, \mathcal{P}_{\mathcal{N}}) \to \mathrm{Hom}(\mathcal{P}_{\mathcal{L}}, \mathcal{P}_{\mathcal{N}})$ for any $\mathcal{L}, \mathcal{M}, \mathcal{N} \in \mathbf{Loc}(X)$ irreducible local systems.

If one can do so, one then knows $\mathrm{End}(\mathcal{P}) \cong \mathrm{A}_p(X) \cong \Bbbk\mathrm{Q}_p(X)/\mathrm{I}_p(X)$. However, the conditions i), ii) and iii) are quite hard to compute in general, therefore this does not help much in practice.

Chapter 5

Indecomposable Perverse Sheaves

In this chapter, we use the equivalences ${}^p\mathbf{Perv}(X) \simeq \mathbf{rep}(\mathrm{Q}_p(X), \mathrm{I}_p(X)) \cong A_p\text{-}\mathbf{mod}$, see Theorem 3.4.0.6, to explain some ideas for helping to compute the Auslander-Reiten quiver of the category ${}^p\mathbf{Perv}(X)$ of p-perverse sheaves on a topologically stratified space X with finitely many strata S, each with finite fundamental group, where $\Bbbk$ is an algebraically closed field of characteristic not dividing the order of $\pi_1(S)$ for any stratum $S \subset X$. Note that this is equivalent to determine the Auslander-Reiten quiver of the finite dimensional algebra $\mathrm{A}_p \cong \mathrm{End}(\mathcal{P})$, where $\mathcal{P} \in {}^p\mathbf{Perv}(X)$ is a projective generator of the category ${}^p\mathbf{Perv}(X)$, see 3.4.0.3. In order to do so, we need to understand indecomposable perverse sheaves and irreducible morphisms between them, see Definition 2.2.3.25 and Remark 2.2.3.26. Our approach is based on adding inductively one closed stratum S at a time.

Section 5.1 is devoted to the study of indecomposable objects. We introduce the notion of small object, which under the assumption that $\pi_1(S)$ is finite, is equivalent to asking that the considered object has no summand supported on S. We then introduce two projection functors, $P_!$ and P_*, and we study their features. Those functors turn out to be very important in order to give a characterisation of indecomposable extensions $\mathcal{E} \in {}^p\mathbf{Perv}(X)$ of a fixed perverse sheaf $\mathcal{F} \in {}^p\mathbf{Perv}(U)$. We give a criterion for the indecomposability of such extensions in terms of the smallness of the object $\mathcal{E} \in {}^p\mathbf{Perv}(X)$ and the splitting of the natural morphism $P_! \to P_*$.

In Section 5.2 we introduce nearby perversities, that is perversities that one can reach by varying the value of a starting perversity by one on a closed union of strata. We point out that tilting at a certain torsion pair is equivalent to changing perversity to a nearby one. We then study how simple objects, projective covers and injective hulls change when one considers a nearby perversity. This gives a way to check if a nearby perversity heart of a faithful one is also faithful or not.

Throughout the chapter, the case of $X = \mathbb{P}^1$ stratified by a point and the open complement, see Example 2.3.1.6.iii), is used as example to check the results.

5.1 Indecomposable Extensions

In this section, we assume that X is a topologically stratified space with finitely many strata S, each with finite fundamental group, p is a perversity on X and $\Bbbk$ an algebraically closed field with characteristic not dividing the order of the group $\pi_1(S)$ for any stratum $S \subset X$. Let $i : S \hookrightarrow X$ denote the inclusion of a closed stratum and $j : U = X \setminus S \hookrightarrow X$ the complementary open inclusion. Let us fix a perverse sheaf $\mathcal{F} \in {}^p\mathbf{Perv}(U)$. We want to characterise indecomposable extensions $\mathcal{E}$ of $\mathcal{F}$, that is indecomposable perverse sheaves $\mathcal{E} \in {}^p\mathbf{Perv}(X)$ such that $j^*\mathcal{E} \cong \mathcal{F}$, see Definition 2.3.5.2.

5.1.1 Small Extensions

Recall that the intermediate restriction functor, see Definition 2.3.5.11, is defined as the image of the natural morphism ${}^pi^! \to {}^pi^*$ in ${}^p\mathbf{Perv}(X)$, that is ${}^pi^{!*} = \mathrm{im}({}^pi^! \to {}^pi^*) : {}^p\mathbf{Perv}(X) \to {}^p\mathbf{Perv}(S)$. We now introduce a class of perverse sheaves which will turn out to be very important in order to detect indecomposable extensions.

Definition 5.1.1.1. *A perverse sheaf $\mathcal{E} \in {}^p\mathbf{Perv}(X)$ is said to be* **small** *if the intermediate restriction functor applied to $\mathcal{E}$ is zero, that is ${}^pi^{!*}\mathcal{E} := \mathrm{im}({}^pi^!\mathcal{E} \to {}^pi^*\mathcal{E}) = 0$.*

We know give an equivalent characterisation of a small object $\mathcal{E} \in {}^p\mathbf{Perv}(X)$ in terms of the summand supported on the closed stratum $S \subset X$.

Lemma 5.1.1.2. *Let X be a topologically stratified space with finitely many strata, each with finite fundamental group. Let us consider a closed stratum S and complementary open and closed inclusions $j : U = X \setminus S \hookrightarrow X$ and $i : S \hookrightarrow X$. Then, $\mathcal{E} \in {}^p\mathbf{Perv}(X)$ is small if and only if it has no summand supported on S.*

Proof. Let us suppose $\mathcal{E} \in {}^p\mathbf{Perv}(X)$ small, that is ${}^pi^{!*}\mathcal{E} \cong 0$. By Lemma 2.3.6.2 we have that $i_*{}^pi^!\mathcal{E}$ and $i_*{}^pi^*\mathcal{E}$ are the maximal quotient and maximal sub-object supported on S

respectively. Then, the diagram

$$\begin{tikzcd} & \mathcal{E} \arrow[dr, two heads] & \\ i_*{}^p i^! \mathcal{E} \arrow[ur, hook] \arrow[rr, "0"'] & & i_*{}^p i^* \mathcal{E} \end{tikzcd}$$

shows that $\mathcal{E}$ cannot have summands in ${}^p\mathbf{Perv}(S)$. Now suppose that $\mathcal{E} \in {}^p\mathbf{Perv}(X)$ has no summand supported on S. Let us consider the diagram

$$\begin{tikzcd} & \mathcal{E} \arrow[dr, two heads] & \\ i_*{}^p i^! \mathcal{E} \arrow[ur, hook] \arrow[rr, "\phi"] \arrow[dr, two heads] & & i_*{}^p i^* \mathcal{E} \\ & i_*{}^p i^{!*} \mathcal{E} \arrow[ur, hook] & \end{tikzcd} .$$

Since $\pi_1(S)$ is finite, then ${}^p\mathbf{Perv}(S)$ is semisimple and using that i_* is exact, we have $i_*{}^p i^{!*}\mathcal{E} \hookrightarrow i_*{}^p i^! \mathcal{E}$. Moreover, using again that ${}^p\mathbf{Perv}(S)$ is semisimple, we have that $i_*{}^p i^! \mathcal{E} \cong \ker\phi \oplus \mathrm{im}\phi$ and $i_*{}^p i^*\mathcal{E} \cong \mathrm{im}\phi \oplus \mathrm{coker}\phi$ with ϕ conjugate to $\begin{pmatrix} 0 & \mathrm{id}_{\mathrm{im}\phi} \\ 0 & 0 \end{pmatrix}$. Therefore, we have $\mathcal{E} \cong i_*{}^p i^{!*}\mathcal{E} \oplus \mathcal{E}'$, but the hypothesis forces $i_*{}^p i^! \mathcal{E} \cong 0$. Then ${}^p i^{!*}\mathcal{E} \cong 0$, hence $\mathcal{E}$ is small. $\square$

There are some extensions of a fixed perverse sheaf $\mathcal{F} \in {}^p\mathbf{Perv}(U)$ which are always small in ${}^p\mathbf{Perv}(X)$.

Remark 5.1.1.3. *Let $\mathcal{F} \in {}^p\mathbf{Perv}(U)$ be a fixed perverse sheaf, the extensions ${}^p j_! \mathcal{F}, {}^p j_* \mathcal{F} \in {}^p\mathbf{Perv}(X)$ are small since by adjunction, see Section 2.3.5, ${}^p i^{*p} j_! \mathcal{F} = {}^p i^{!p} j_* \mathcal{F} \cong 0$. The intermediate extension ${}^p j_{!*}\mathcal{F}$ is also small as it has no non-zero sub-object or quotient supported on S, see Remark 2.3.5.5. Moreover, Beilinson's maximal extension $\mathcal{M}$ of a perverse sheaf $\mathcal{F} \in {}^p\mathbf{Perv}(U)$, when it exists, see Section 2.3.5, is small. Indeed, let consider the diagram*

$$\begin{tikzcd} & \mathcal{M} \arrow[dr, two heads] & \\ i_*{}^p i^! \mathcal{M} \arrow[ur, hook] \arrow[rr, "\alpha"] & & i_*{}^p i^* \mathcal{M} \end{tikzcd}$$

If $\alpha \neq 0$, then by [Rei10, pag.19] it is an isomorphism, therefore $\mathcal{M} \cong {}^p i^! \mathcal{F} \oplus \mathcal{F}'$. This is *impossible as $\mathcal{M} \in {}^p\mathbf{Perv}(X)$ is indecomposable, thus $\alpha = 0$. This implies that ${}^p i^{!*}\mathcal{M} \cong 0$, that is $\mathcal{M}$ is small.*

Example 5.1.1.4. *Let us consider $X = \mathbb{P}^1$ stratified as in Example 2.3.1.6.iii) with the middle perversity. Then, all the four extensions of the objects $\Bbbk_U[1] \in {}^m\mathbf{Perv}(U)$, namely $j_!\Bbbk_U[1], \Bbbk_X[1], j_*\Bbbk_U[1], \mathcal{M} \in {}^m\mathbf{Perv}(X)$, are small.*

We now study which classes of morphisms are preserved under ${}^p i^!, {}^p i^*$ and ${}^p i^{!*}$.

Lemma 5.1.1.5. *The functors ${}^p i^!$ and ${}^p i^*$ preserve monomorphisms and epimorphisms between objects in ${}^p\mathbf{Perv}(X)$ which are extensions of a fixed perverse sheaf $\mathcal{F} \in {}^p\mathbf{Perv}(U)$.*

Proof. First note that the two statements are dual to each other, therefore we prove the first claim. Let $\mathcal{G}, \mathcal{H} \in {}^p\mathbf{Perv}(X)$ be extensions of a fixed $\mathcal{F} \in {}^p\mathbf{Perv}(U)$ and consider a morphism $\alpha \in \mathrm{Hom}_{{}^p\mathbf{Perv}(X)}(\mathcal{G}, \mathcal{H})$. We have two triangles

$$i^!\mathcal{G} \to \mathcal{G} \to j_*j^*\mathcal{G} \cong j_*\mathcal{F}$$
$$i^!\mathcal{H} \to \mathcal{H} \to j_*j^*\mathcal{H} \cong j_*\mathcal{F}$$

in $\mathbf{D}_{\mathrm{c}}(X)$. Considering perverse cohomology gives two long exact sequences and in particular a diagram

$$\begin{tikzcd}
0 \cong {}^pH^{-1}(j_*\mathcal{F}) \arrow[r, equal] \arrow[d] & {}^pH^{-1}(j_*\mathcal{F}) \cong 0 \arrow[d] \\
{}^p i^!\mathcal{G} \arrow[r, "{}^p i^!\alpha"] \arrow[d, hook, "i_\mathcal{G}"'] & {}^p i^!\mathcal{H} \arrow[d, hook, "i_\mathcal{H}"] \\
\mathcal{G} \arrow[r, "\alpha"] \arrow[d] & \mathcal{H} \arrow[d] \\
{}^p j_*\mathcal{F} \arrow[r, equal] \arrow[d] & {}^p j_*\mathcal{F} \arrow[d] \\
{}^pH^1({}^p i^!\mathcal{G}) \arrow[r] \arrow[d] & {}^pH^1({}^p i^!\mathcal{H}) \arrow[d] \\
\cdots & \cdots
\end{tikzcd}$$

Now, if α is monic, then so is $\alpha \circ i_\mathcal{G} = i_\mathcal{H} \circ {}^p i^!\alpha$ hence ${}^p i^!\alpha$ is monic as well. On the other hand, if α is epic by diagram chasing also ${}^p i^!\alpha$ is epic. $\square$

Corollary 5.1.1.6. *The intermediate restriction functor* ${}^pi^{!*}$ *preserves monomorphisms and epimorphisms between objects in* ${}^p\mathbf{Perv}(X)$ *which are extensions of a fixed perverse sheaf* $\mathcal{F} \in {}^p\mathbf{Perv}(U)$.

Proof. The claims follows from the definition of intermediate restriction functor and Lemma 5.1.1.5. ☐

Remark 5.1.1.7. *The situation described in Lemma 5.1.1.5, and Corollary 5.1.1.6, is not necessarily true for morphisms between objects which are extensions of different objects in* ${}^p\mathbf{Perv}(U)$.

We now show that being small is a property inherited by sub-objects and quotients, which justifies the name.

Proposition 5.1.1.8. *Sub-objects and quotients of small objects in* ${}^p\mathbf{Perv}(X)$ *which are extensions of a fixed perverse sheaf* ${}^p\mathbf{Perv}(U)$ *are small.*

Proof. Note that the two statements are dual, hence we prove only the first one. Let $\mathcal{E} \in {}^p\mathbf{Perv}(X)$ be a small extension of a fixed $\mathcal{F} \in {}^p\mathbf{Perv}(U)$, that is

$$\begin{tikzcd}
{}^pi^!\mathcal{E} \arrow[rr, "\alpha"] \arrow[dr, two heads] & & {}^pi^*\mathcal{E} \\
& {}^pi^{!*}\mathcal{E} \cong 0 \arrow[ur, hook] &
\end{tikzcd}$$

Let us consider a sub-object $\mathcal{E}' \in {}^p\mathbf{Perv}(X)$ of $\mathcal{E}$ of a fixed $\mathcal{F} \in {}^p\mathbf{Perv}(U)$, that is there is a monomorphism $i_{\mathcal{E}'} : \mathcal{E}' \hookrightarrow \mathcal{E}$ in ${}^p\mathbf{Perv}(X)$ and $j^*\mathcal{E}' \cong \mathcal{F}$. We have a diagram

$$\begin{tikzcd}
{}^pi^!\mathcal{E}' \arrow[rr] \arrow[d, hook] & & {}^pi^*\mathcal{E}' \arrow[d, hook] \\
{}^pi^!\mathcal{E} \arrow[rr, "\alpha"] \arrow[dr, two heads] & & {}^pi^*\mathcal{E} \\
& 0 \arrow[ur, hook] &
\end{tikzcd}$$

where the vertical arrows are monomorphisms by Lemma 5.1.1.5. Therefore, the map ${}^pi^!\mathcal{E}' \to {}^pi^*\mathcal{E}'$ is zero. Hence ${}^pi^{!*}\mathcal{E}' = 0$, that is $\mathcal{E}' \in {}^p\mathbf{Perv}(X)$ is small. ☐

5.1.2 Projection Functors

We start this section with a general result on the image of a natural transformation. This result allows us to introduce two functors which, as the intermediate extension, see Definition 2.3.5.4, and intermediate restriction functors, see Definition 2.3.5.11, arise as the image of a natural transformation between functors. We then study some properties of such functors as they will play an important role later.

Lemma 5.1.2.1. *Let $\mathcal{A}$ be any category and $\mathcal{B}$ an abelian category. Let us consider functors $F, G : \mathcal{A} \to \mathcal{B}$ and a natural transformation $\eta : F \to G$. Then $\mathrm{im}\eta$ is a functor.*

Proof. For every $A \in \mathcal{A}$ the component of η at A is given by $\eta_A : F(A) \to G(A)$, therefore $(\mathrm{im}\eta)(A) = \mathrm{im}\eta_A$ is a morphism in $\mathcal{B}$. For any $A, A' \in \mathcal{A}$ and for any $\alpha \in \mathrm{Hom}_{\mathcal{A}}(A, A')$ there is a diagram

$$\begin{tikzcd}
F(A) \arrow[rr, "\eta_A"] \arrow[rdd, phantom] \arrow[dddd, "F(\alpha)"'] \arrow[rd, two heads] & & G(A) \arrow[dddd, "G(\alpha)"] \\
& \mathrm{im}\eta_A \arrow[ru, hook] \arrow[d, dashed] & \\
& \mathrm{im}\eta_{A'} \arrow[rd, hook] & \\
F(A') \arrow[ru, two heads] \arrow[rr, "\eta_{A'}"'] & & G(A')
\end{tikzcd}$$

which shows that there exists a map $\mathrm{im}\eta_A \to \mathrm{im}\eta_{A'}$. In order to check the functoriality of this construction, we note that $(\mathrm{im}\eta)(\mathrm{id}_A) = \mathrm{id}_{(\mathrm{im}\eta)(A)}$. Moreover, from the diagram

$$\begin{tikzcd}
F(A) \arrow[r, two heads] \arrow[d] & \mathrm{im}\eta_A \arrow[r, hook] \arrow[d, dashed] \arrow[dd, dashed, bend left] & G(A) \arrow[d] \\
F(A') \arrow[r, two heads] \arrow[d] & \mathrm{im}\eta_{A'} \arrow[r, hook] \arrow[d, dashed] & G(A') \arrow[d] \\
F(A'') \arrow[r, two heads] & \mathrm{im}\eta_{A''} \arrow[r, hook] & G(A'')
\end{tikzcd}$$

it follows that

$$(\mathrm{im}\eta)(\alpha' \circ \alpha) = (\mathrm{im}\eta)(\alpha') \circ (\mathrm{im}\eta)(\alpha).$$

$\square$

Remark 5.1.2.2. *Note that Lemma 5.1.2.1 confirms that the intermediate extension*

${}^p j_{!*} = \mathrm{im}({}^p j_! \to {}^p j_*)$ *and the intermediate restriction* ${}^p i^{!*} = \mathrm{im}({}^p i^! \to {}^p i^*)$ *are well-defined functors.*

We can now define two additive **projection functors**

$$\begin{aligned} P_! &:= \mathrm{im}({}^p j_! j^* \to \mathrm{id}) : {}^p\mathbf{Perv}(X) \to {}^p\mathbf{Perv}(X) \\ P_* &:= \mathrm{im}(\mathrm{id} \to {}^p j_* j^*)) : {}^p\mathbf{Perv}(X) \to {}^p\mathbf{Perv}(X). \end{aligned} \tag{5.1}$$

Note that Lemma 5.1.2.1 ensures that the functors in (5.1) are well defined. Moreover, there is a canonical morphism

$$\begin{tikzcd} P_!\mathcal{E} \arrow[rr, "\beta\circ\alpha"] \arrow[dr, twoheadrightarrow, "\alpha"'] & & P_*\mathcal{E} \\ & {}^p j_{!*}\mathcal{F} \arrow[ur, hookrightarrow, "\beta"'] & \end{tikzcd} \tag{5.2}$$

We now show that for $\mathcal{E} \in {}^p\mathbf{Perv}(X)$ the objects $P_!\mathcal{E}$ and $P_*\mathcal{E}$ sit in some particular short exact sequences. We use this characterisation to give an interpretation of $P_!\mathcal{E}, P_*\mathcal{E} \in {}^p\mathbf{Perv}(X)$ in terms of sub-objects and quotients supported on the closed stratum S.

Lemma 5.1.2.3. *Let* $\mathcal{E} \in {}^p\mathbf{Perv}(X)$. *Then in* ${}^p\mathbf{Perv}(X)$ *there are short exact sequences*

$$0 \to P_!\mathcal{E} \to \mathcal{E} \to {}^p i^*\mathcal{E} \to 0$$
$$0 \to {}^p i^!\mathcal{E} \to \mathcal{E} \to P_*\mathcal{E} \to 0.$$

Proof. Note that the two short exact sequence are dual to each other. Therefore we prove the first statement. Let us consider the triangle

$$j_! j^*\mathcal{E} \to \mathcal{E} \to i^*\mathcal{E} \to j_! j^*\mathcal{E}[1]$$

in $\mathbf{D}_{\mathrm{c}}(X)$. Taking perverse cohomology yields a long exact sequence

$$\ldots \to {}^pH^{-1}(i^*\mathcal{E}) \to {}^p j_! j^*\mathcal{E} \to \mathcal{E} \to {}^p i^*\mathcal{E} \to {}^pH^1(j_! j^*\mathcal{E}) \to \ldots$$

Since $j_!$ is right t-exact, then ${}^pH^1(j_! j^*\mathcal{E}) = 0$. Therefore the above long exact sequence becomes

$$
\begin{tikzcd}
\dots \arrow[r] & {}^pH^{-1}(i^*\mathcal{E}) \arrow[r] & {}^pj_!j^*\mathcal{E} \arrow[rr] \arrow[dr, two heads] & & \mathcal{E} \arrow[r, two heads] & {}^pi^*\mathcal{E} \arrow[r] & 0 \\
& & & P_!\mathcal{E} \arrow[ur, hook] & & &
\end{tikzcd}
$$

which gives the first of the two short exact sequences. $\square$

Remark 5.1.2.4. *Given an object $\mathcal{E} \in {}^p\mathbf{Perv}(X)$ which is an extension of a fixed perverse sheaf $\mathcal{F} \in {}^p\mathbf{Perv}(U)$, the two short exact sequences of Lemma 5.1.2.3 give the following characterisation of the projection functors $P_!, P_*$ defined in (5.1):*

i) $P_!\mathcal{E}$ is the minimal sub-object of $\mathcal{E}$ such that the quotient is supported on S.

ii) $P_*\mathcal{E}$ is the minimal quotient of $\mathcal{E}$ such that the kernel is supported on S.

Moreover, by adjunction ${}^pi^{*p}j_! = {}^pi^{!p}j_* = 0$ and we have ${}^pi^*P_!\mathcal{E} = {}^pi^!P_*\mathcal{E} = 0$, that is the projection functors $P_!$ and P_* take perverse sheaves in ${}^p\mathbf{Perv}(X)$ which are extensions of a fixed perverse sheaf $\mathcal{F} \in {}^p\mathbf{Perv}(U)$ to small perverse sheaves with no quotients and no sub-objects supported on S respectively.

We now study how morphisms in ${}^p\mathbf{Perv}(X)$ between extensions of a fixed perverse sheaf $\mathcal{F} \in {}^p\mathbf{Perv}(X)$, that is such that when restricted to ${}^p\mathbf{Perv}(U)$ are the identity on the object $\mathcal{F} \in {}^p\mathbf{Perv}(U)$, behave under the projection functors $P_!$ and P_* defined in (5.1).

Lemma 5.1.2.5. *Let $\mathcal{G}, \mathcal{H} \in {}^p\mathbf{Perv}(X)$ be extensions of a fixed perverse sheaf $\mathcal{F} \in {}^p\mathbf{Perv}(U)$. Let us consider a morphism $\alpha \in \mathrm{Hom}_{{}^p\mathbf{Perv}(X)}(\mathcal{G}, \mathcal{H})$ such that $j^*\alpha = \mathrm{id}_{\mathcal{F}}$.*

i) If α is a monomorphism, then $P_!\alpha$ is an isomorphism.

ii) If α is an epimorphism, then $P_*\alpha$ is an isomorphism.

Proof. Note that the two statements are dual, therefore we prove only the first one. We have a diagram

$$\begin{tikzcd}
{}^p j_! j^* \mathcal{G} \arrow[rr] \arrow[dr, two heads, "p_1"] \arrow[ddd, equal, "j^*\alpha = \mathrm{id}"'] & & \mathcal{G} \arrow[ddd, hook, "\alpha"] \\
& P_! \mathcal{G} \arrow[ur, hook, "i_1"'] \arrow[d, dashed, "P_! \alpha"] & \\
& P_! \mathcal{H} \arrow[dr, hook, "i_2"] & \\
{}^p j_! j^* \mathcal{G} \arrow[rr] \arrow[ur, two heads, "p_2"] & & \mathcal{H}
\end{tikzcd}$$

In particular, we have that

$$P_! \alpha \circ p_1 = p_2 \circ j^* \alpha = p_2$$

is an epimorphism, then so is $P_! \alpha$. Note that such conclusion holds without any assumption on α. If in addition we assume that α is monic, we have that

$$\alpha \circ i_1 = i_2 \circ P_! \alpha$$

is monic and so is $P_! \alpha$. Therefore $P_! \alpha$ is an isomorphism. □

Remark 5.1.2.6. *Let* $\alpha \in \mathrm{Hom}_{{}^p\mathbf{Perv}(X)}(\mathcal{G}, \mathcal{H})$ *such that* $j^* \alpha = \mathrm{id}_{\mathcal{F}}$ *for a certain fixed perverse sheaf* $\mathcal{F} \in {}^p\mathbf{Perv}(U)$*. The proof of Lemma 5.1.2.5 implies that* $P_! \alpha$ *is always an epimorphism while* $P_* \alpha$ *is always a monomorphism.*

Remark 5.1.2.7. *The converse of Lemma 5.1.2.5 does not hold. Let us consider the projection* $\alpha : \mathcal{G} \oplus \mathcal{H} \to \mathcal{G}$ *in* ${}^p\mathbf{Perv}(X)$*, where* $\mathcal{H} \in {}^p\mathbf{Perv}(S)$ *is a non-zero object. Then* $P_! \alpha$ *is an isomorphism even though* α *is not monic.*

5.1.3 The Ice Cream Cone Picture

In this section, we fix a perverse sheaf $\mathcal{F} \in {}^p\mathbf{Perv}(U)$ and study its extensions over S, that is we consider the category $\mathbb{E}(\mathcal{F})$ with objects $(\mathcal{E}, \phi)$, where $\mathcal{E} \in {}^p\mathbf{Perv}(X)$ with $\phi : j^* \mathcal{E} \to \mathcal{F}$ an isomorphism in ${}^p\mathbf{Perv}(U)$, and morphisms given by $\alpha : \mathcal{E} \to \mathcal{E}'$ such that the diagram

$$\begin{tikzcd}
j^* \mathcal{E} \arrow[r, "j^*\alpha"] \arrow[d, "\phi"] & j^* \mathcal{E}' \arrow[d, "\phi'"] \\
\mathcal{F} \arrow[r, equal] & \mathcal{F}
\end{tikzcd}$$

commutes, that is $\phi' \circ j^*\alpha = \phi$. We show we can characterise initial and final extensions of the fixed object $\mathcal{F} \in {}^p\mathbf{Perv}(U)$ with an isomorphism $\phi : j^*\mathcal{E} \to \mathcal{F}$ in the category $\mathbb{E}(\mathcal{F})$. Moreover, there will be a minimal extension, given by the intermediate extension functor, and a maximal one. We then organise the information about extensions of $\mathcal{F}$ and maps between them and highlight the behaviour of some morphisms.

Proposition 5.1.3.1. *Let $\mathcal{F} \in {}^p\mathbf{Perv}(U)$ be a fixed perverse sheaf.*

i) The extension ${}^pj_!\mathcal{F} \in {}^p\mathbf{Perv}(X)$ is initial in $\mathbb{E}(\mathcal{F})$.

ii) The extension ${}^pj_*\mathcal{F} \in {}^p\mathbf{Perv}(X)$ is final in $\mathbb{E}(\mathcal{F})$.

Proof. Note that the two statements are dual, therefore we prove only the first one. By adjunction, there is a bijection

$$\mathrm{Hom}_{{}^p\mathbf{Perv}(X)}({}^pj_!\mathcal{F}, \mathcal{E}) \cong \mathrm{Hom}_{{}^p\mathbf{Perv}(U)}(\mathcal{F}, j^*\mathcal{E})$$

which sends $\beta : {}^pj_!\mathcal{F} \to \mathcal{E}$ to $j^*\beta : \mathcal{F} \to j^*\mathcal{E}$. Moreover, composing with the isomorphism $\phi : j^*\mathcal{E} \to \mathcal{F}$ gives that $\mathrm{Hom}_{{}^p\mathbf{Perv}(U)}(\mathcal{F}, j^*\mathcal{E}) \cong \mathrm{Hom}_{{}^p\mathbf{Perv}(U)}(\mathcal{F}, \mathcal{F})$. From the diagram

$$\begin{tikzcd} j^{*p}j_!\mathcal{F} \arrow[rr, "j^*\beta"] \arrow[dr, "\mathrm{id}"'] & & j^*\mathcal{E} \arrow[dl, "\phi"] \\ & \mathcal{F} & \end{tikzcd}$$

we have that $j^*\beta = \phi^{-1}$. Thus, for any object $\mathcal{E} \in {}^p\mathbf{Perv}(X)$ there exists $\beta : {}^pj_!\mathcal{F} \to \mathcal{E}$ corresponding to the identity map $\mathrm{id}_{\mathcal{F}} : \mathcal{F} \to \mathcal{F}$ in ${}^p\mathbf{Perv}(U)$. This proves that ${}^pj_!\mathcal{E}$ is initial in $\mathbb{E}(\mathcal{F})$. $\square$

Note that the extensions ${}^pj_!\mathcal{F}, {}^pj_*\mathcal{F} \in {}^p\mathbf{Perv}(X)$, where $\mathcal{F} \in {}^p\mathbf{Perv}(U)$ is fixed and $\phi : j^*\mathcal{E} \to \mathcal{F}$ is an isomorphism, are initial and final respectively in $\mathbb{E}(\mathcal{F})$, but not in ${}^p\mathbf{Perv}(X)$.

We can therefore organise the information of Section 5.1.2 and Proposition 5.1.3.1 about extensions of a fixed perverse sheaf $\mathcal{F} \in {}^p\mathbf{Perv}(U)$ in the following picture.

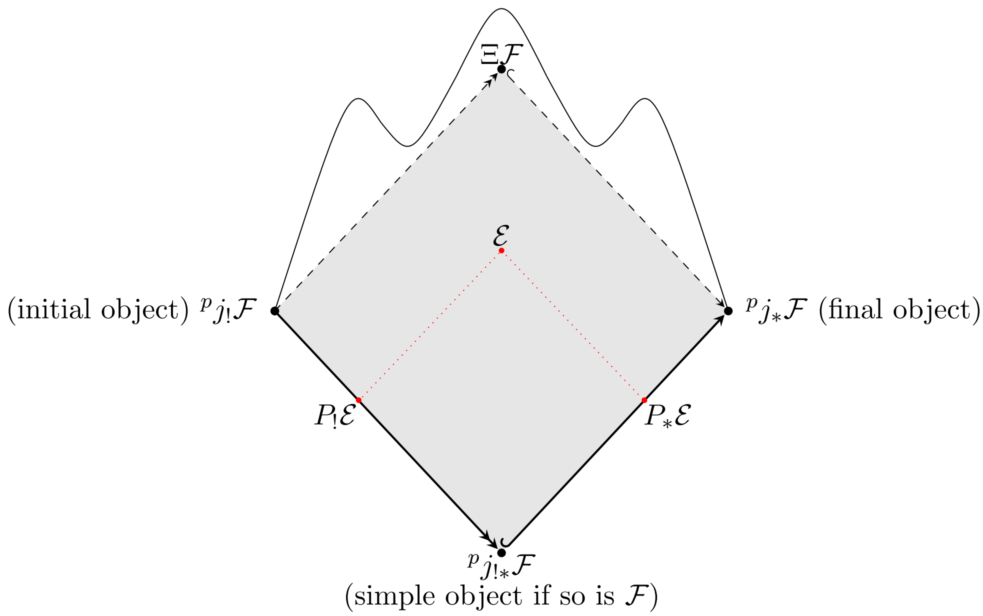

Figure 5.1: Extensions $\mathcal{E} \in {}^p\mathbf{Perv}(X)$ of a fixed perverse sheaf $\mathcal{F} \in {}^p\mathbf{Perv}(U)$.

In Figure 5.1, the object $\Xi\mathcal{F} \in {}^p\mathbf{Perv}(X)$ denotes a maximal extension of $\mathcal{F} \in {}^p\mathbf{Perv}(U)$. Note that if X is an algebraic variety, then $\Xi\mathcal{F}$ coincides with Beilinson's maximal extension, see Section 2.3.5 and [Rei10, Bei87a]. The shaded part indicates the small extensions of $\mathcal{F} \in {}^p\mathbf{Perv}(U)$. The objects which lie on the lower left edge have no quotient supported on S, that is they are in $\ker {}^p i^*$. On the lower right edge lie objects with no sub-object supported on S, that is in $\ker {}^p i^!$.

Now, let $\mathcal{E}, \mathcal{G} \in {}^p\mathbf{Perv}(X)$ be extensions of a fixed perverse sheaf $\mathcal{F} \in {}^p\mathbf{Perv}(X)$. Let us consider a morphism $\alpha \in \mathrm{Hom}_{{}^p\mathbf{Perv}(X)}(\mathcal{E}, \mathcal{G})$ which restricts to the identity on $\mathcal{F}$ on U, that is such that $j^*\alpha = \mathrm{id}_{\mathcal{F}}$. By Lemma 5.1.2.5 the projection functors $P_!$ and P_* of (5.1) behave as follows.

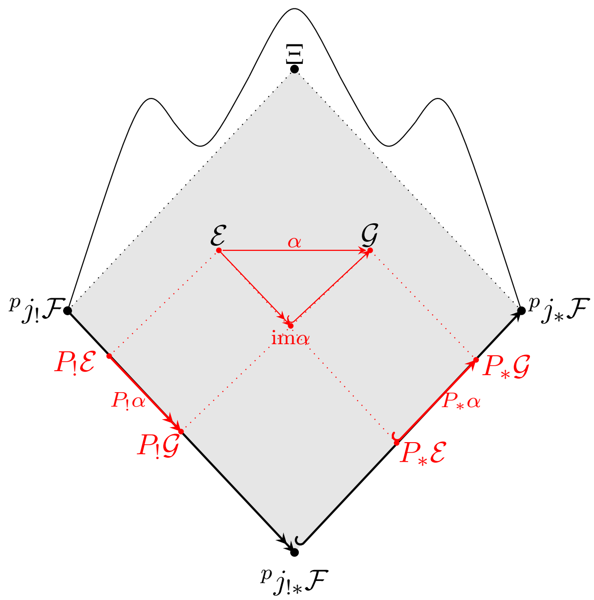

Figure 5.2: Projections of a morphisms between extensions of a fixed $\mathcal{F} \in {}^p\mathbf{Perv}(U)$.

5.1.4 Indecomposable Extensions over a Closed Stratum

In this section, we show that small extensions of a fixed perverse sheaf $\mathcal{F} \in {}^p\mathbf{Perv}(X)$ are in one to one correspondence with pairs of perverse sheaves $(\mathcal{A}, \mathcal{B})$ in ${}^p\mathbf{Perv}(X) \times {}^p\mathbf{Perv}(X)$ such that $\mathcal{A}$ has no sub-object supported on S and $\mathcal{B}$ has no quotient supported on S. We then give a criterion which allows us to check if an extension $\mathcal{E} \in {}^p\mathbf{Perv}(X)$ of $\mathcal{F} \in {}^p\mathbf{Perv}(U)$ is indecompsable. Indeed, we show that an extension $\mathcal{E}$ of a fixed perverse sheaf $\mathcal{F} \in {}^p\mathbf{Perv}(U)$ is indecomposable if and only if $\mathcal{E}$ is small and the natural morphism $\beta \circ \alpha : P_!\mathcal{E} \to P_*\mathcal{E}$ does not split.

Suppose that $\mathcal{E} \in {}^p\mathbf{Perv}(X)$ is a small extension of $\mathcal{F} \in {}^p\mathbf{Perv}(U)$. Then, Lemma 5.1.2.3 gives two short exact sequences in ${}^p\mathbf{Perv}(X)$ that we can use to construct the following commutative diagram.

$$
\begin{tikzcd}
 & {}^p i^! \mathcal{E} \arrow[rr, "0"] \arrow[dr] & & {}^p i^* \mathcal{E} \arrow[dr, equal] & \\
{}^p i^! \mathcal{E} \arrow[ur, equal] \arrow[dr] & & \mathcal{E} \arrow[ur] \arrow[dr] & & {}^p i^* \mathcal{E} \\
 & P_! \mathcal{E} \arrow[ur] \arrow[dr] & & P_* \mathcal{E} \arrow[ur] \arrow[dr] & \\
{}^p i^* \mathcal{E}[-1] \arrow[ur] \arrow[dr, equal] & & {}^p j_{!*} \mathcal{F} \arrow[ur] \arrow[dr] & & {}^p i^! \mathcal{E}[1] \\
 & {}^p i^* \mathcal{E}[-1] \arrow[rr, "0"'] \arrow[ur] & & {}^p i^! \mathcal{E}[1] \arrow[ur, equal] &
\end{tikzcd}
\tag{5.3}
$$

Note that the diagonals are exact triangles in $\mathbf{D}_c(X)$.

Definition 5.1.4.1. *The pair* $(\mathcal{A}, \mathcal{B}) \in {}^p\mathbf{Perv}(X) \times {}^p\mathbf{Perv}(X)$ *is an* **extension pair relative to** $\mathcal{F} \in {}^p\mathbf{Perv}(U)$ *if:*

i) $j^*\mathcal{A} \cong \mathcal{F} \cong j^*\mathcal{B}$.

ii) ${}^p i^* \mathcal{A} \cong 0 \cong {}^p i^! \mathcal{B}$.

iii) The class $\alpha\beta = 0 \in \mathrm{Hom}_{\mathbf{D}_c(X)}({}^p i^* \mathcal{B}[-1], {}^p i^! \mathcal{A}[1]) \cong \mathrm{Ext}^2_{\mathbf{D}_c(X)}({}^p i^* \mathcal{B}, {}^p i^! \mathcal{A})$,

where in the third condition the classes $\alpha \in \mathrm{Hom}_{\mathbf{D}_c(X)}({}^p j_{!*} \mathcal{F}, {}^p i^! \mathcal{A}[1])$ *and* $\beta \in \mathrm{Hom}_{\mathbf{D}_c(X)}({}^p i^* \mathcal{B}[-1], {}^p j_{!*} \mathcal{F})$ *are those classifying the short exact sequences* $0 \to {}^p i^! \mathcal{A} \to \mathcal{A} \to {}^p j_{!*} \mathcal{F} \to 0$ *and* $0 \to {}^p j_{!*} \mathcal{F} \to \mathcal{B} \to {}^p i^* \mathcal{B} \to 0$*, which exist by conditions i) and ii).*

Remark 5.1.4.2. *If* $\mathcal{E} \in {}^p\mathbf{Perv}(X)$ *is a small extension of* $\mathcal{F} \in {}^p\mathbf{Perv}(U)$*, then by (5.3)* $(P_! \mathcal{E}, P_* \mathcal{E})$ *is an extension pair relative to* $\mathcal{F}$*.*

We now show that there is a very strong relationship between small extensions and extension pairs.

Theorem 5.1.4.3. *Let* $\mathcal{F} \in {}^p\mathbf{Perv}(U)$ *be a fixed perverse sheaf. There is a bijection*

$$
\left\{ \begin{matrix} \textit{Small Extensions } \mathcal{E} \textit{ of} \\ \mathcal{F} \in {}^p\mathbf{Perv}(U) \end{matrix} \right\}_{/\cong} \longleftrightarrow \left\{ \begin{matrix} \textit{Extension Pairs } (\mathcal{A}, \mathcal{B}) \\ \textit{relative to } \mathcal{F} \in {}^p\mathbf{Perv}(U) \end{matrix} \right\}_{/\cong}.
$$

Proof. We have already seen in Remark 5.1.4.2 that if $\mathcal{E} \in {}^p\mathbf{Perv}(X)$ is a small extension of $\mathcal{F} \in {}^p\mathbf{Perv}(U)$, then $(P_!\mathcal{E}, P_*\mathcal{E})$ is an extension pair relative to $\mathcal{F}$. Since $P_!$ and P_* are functors, this descends to a map on isomorphism classes.

Now suppose that $(\mathcal{A}, \mathcal{B})$ is an extension pair. As noted above, there are triangles

$$\begin{aligned} &{}^p i^! \mathcal{A} \to \mathcal{A} \to {}^p j_{!*}\mathcal{F} \xrightarrow{\alpha} {}^p i^! \mathcal{A}[1] \\ &{}^p i^* \mathcal{B}[-1] \xrightarrow{\beta} {}^p j_{!*}\mathcal{F} \to \mathcal{B} \to {}^p i^* \mathcal{B} \end{aligned} \tag{5.4}$$

in $\mathbf{D}_c(X)$, where $\alpha\beta = 0$. Since $\pi_1(S)$ is finite the category ${}^p\mathbf{Perv}(X)$ is semisimple. Hence, we have that $\mathrm{Hom}({}^p i^* \mathcal{B}[-1], {}^p i^! \mathcal{A}) \cong 0$ so that the first triangle in (5.4) induces an exact sequence

$$0 \to \mathrm{Hom}({}^p i^* \mathcal{B}[-1], \mathcal{A}) \to \mathrm{Hom}({}^p i^* \mathcal{B}[-1], {}^p j_{!*}\mathcal{F}) \xrightarrow{\alpha_\cdot} \mathrm{Hom}({}^p i^* \mathcal{B}[-1], {}^p i^! \mathcal{A}[1]) \to \dots$$

Since $\alpha\beta = 0$, there is a unique factorisation of β via $\tilde{\beta} \in \mathrm{Hom}({}^p i^* \mathcal{B}[-1], \mathcal{A})$. We define $\mathcal{E}(\mathcal{A}, \mathcal{B}) = \mathrm{cone}(\tilde{\beta})$. By construction, we have that $\mathcal{E}(\mathcal{A}, \mathcal{B}) \in {}^p\mathbf{Perv}(X)$ is an extension of $\mathcal{F}$. One can verify that, up to isomorphism, $\mathcal{E}(\mathcal{A}, \mathcal{B})$ depends only on the isomorphism class of $(\mathcal{A}, \mathcal{B})$. Moreover, the octahedral axiom, see Definition 2.1.3.3.TR5), applied to the factorisation of β guarantees that there is an octahedron

$$\begin{tikzcd}
{}^p i^* \mathcal{B}[-1] \arrow[rr, "\tilde{\beta}"] \arrow[rrrr, bend left, "\beta"] & & \mathcal{A} \arrow[rr] \arrow[dl] & & {}^p j_{!*}\mathcal{F} \arrow[dl] \arrow[ddll, bend left] \\
& \mathcal{E}(\mathcal{A}, \mathcal{B}) \arrow[ul, dashed] \arrow[dr] & & {}^p i^! \mathcal{A}[1] \arrow[ll, dashed] \arrow[ul, dashed] & \\
& & \mathcal{B} \arrow[ur] \arrow[uull, dashed, bend left] & &
\end{tikzcd} \tag{5.5}$$

where the dashed arrows denote the shift by [1], the triangles with odd numbers of dashed arrows are exact and those with even number of dashed arrows are commutative. It follows that there is a short exact sequence of perverse sheaves

$$0 \to \mathcal{A} \to \mathcal{E}(\mathcal{A}, \mathcal{B}) \to {}^p i^* \mathcal{B} \to 0. \tag{5.6}$$

By applying ${}^p i^*$ to the above short exact sequence we deduce that ${}^p i^* \mathcal{E}(\mathcal{A}, \mathcal{B}) \cong {}^p i^* \mathcal{B}$.

Hence $\mathcal{A} \cong P_!\mathcal{E}(\mathcal{A},\mathcal{B})$ by Lemma 5.1.2.3. Similarly, there is a short exact sequence

$$0 \to {}^p i^! \mathcal{A} \to \mathcal{E}(\mathcal{A},\mathcal{B}) \to \mathcal{B} \to 0$$

which implies that ${}^p i^! \mathcal{E}(\mathcal{A},\mathcal{B}) \cong {}^p i^! \mathcal{A}$ and hence $\mathcal{B} \cong P_*\mathcal{E}(\mathcal{A},\mathcal{B})$. It follows from (5.5), for instance by applying the functor $\mathrm{Hom}({}^p i^! \mathcal{A}, -)$ to (5.6), that $\mathcal{E}(\mathcal{A},\mathcal{B})$ is a small extension of $\mathcal{F}$ and with (5.3) we verify that $\mathcal{E} \cong \mathcal{E}(P_!\mathcal{E}, P_*\mathcal{E})$. Since we have also shown that $P_!\mathcal{E}(\mathcal{A},\mathcal{B}) \cong \mathcal{A}$ and $P_*\mathcal{E}(\mathcal{A},\mathcal{B}) \cong \mathcal{B}$, we are done. $\square$

Remark 5.1.4.4. *Theorem 5.1.4.3 gives 'coordinates' on small extensions $\mathcal{E} \in {}^p\mathbf{Perv}(X)$ of a fixed perverse sheaf $\mathcal{F} \in {}^p\mathbf{Perv}(U)$. That is, in order to understand small extensions $\mathcal{E}$ of $\mathcal{F}$, it is enough to study objects of the form $P_!\mathcal{G}$ and $P_*\mathcal{G}$ for some $\mathcal{G} \in {}^p\mathbf{Perv}(X)$ extension of $\mathcal{F}$.*

Remark 5.1.4.5. *If X is a topologically stratified space with finitely many strata S, each with finite fundamental group, Theorem 5.1.4.3 allows us to define the maximal extension. In fact, let $\mathcal{F} \in {}^p\mathbf{Perv}(U)$ then the object corresponding to the pair $({}^p j_!\mathcal{F}, {}^p j_*\mathcal{F})$ is its maximal extension. Note that this agrees with Beilinson's maximal extension $\Xi\mathcal{F}$ when $\Xi\mathcal{F}$ exists.*

We now show through an example, that there are some quivers with relations (Q,I) which cannot arise as a category of p-perverse sheaves on a topologically stratified space X.

Example 5.1.4.6. *Let us consider the quiver*

$$\mathrm{Q} = \begin{tikzcd} 1 \arrow[r, bend left, "a"] & 0 \arrow[l, bend left, "b"] \end{tikzcd}$$

with relations $ab = 0 = ba$. It is easy to note that there are four indecomposable representations in $\mathbf{rep}(\mathrm{Q},\mathrm{I})$. More specifically, there are two simple and two projective-injective objects in $\mathbf{rep}(\mathrm{Q},\mathrm{I})$ as in the following table.

Object	**Quiver Representation**
S_0	$0 \rightleftarrows \Bbbk$
S_1	$\Bbbk \rightleftarrows 0$
$P_0 \cong I_1$	$\Bbbk \underset{1}{\rightleftarrows} \Bbbk$
$P_1 \cong I_0$	$\Bbbk \overset{1}{\rightleftarrows} \Bbbk$

Table 5.1: Indecomposable representations in $\mathbf{rep}(\mathrm{Q},\mathrm{I})$.

The Auslander-Reiten quiver of $\mathbf{rep}(\mathrm{Q},\mathrm{I})$ *is*

$$\begin{tikzcd}
 & & P_1 \cong I_0 \arrow[dr, two heads] & & P_0 \cong I_1 \arrow[dr, two heads] & & \\
= & S_0 \arrow[ur, hook] & & S_1 \arrow[ll, dashed] \arrow[ur, hook] & & S_0 \arrow[ll, dashed] & =
\end{tikzcd}$$

Theorem 5.1.4.3 (and Remark 5.1.4.5) says that there should exist an indecomposable extension, namely the maximal one, in correspondence with the pair (P_1, P_0)*. Such extension is forced to be decomposable. Therefore, there is no topologically stratified space for which there is an equivalence of categories* $\mathbf{rep}(\mathrm{Q},\mathrm{I}) \simeq {}^p\mathbf{Perv}(X)$*.*

Remark 5.1.4.7. *Let* $\mathcal{E} \cong E(\beta \circ \alpha)$ *be the (isomorphism) class of the extension built in Theorem 5.1.4.3 relative to the canonical morphism* $\beta \circ \alpha : P_! \mathcal{E} \to P_* \mathcal{E}$*. If such canonical morphism splits, that is if* $\beta \circ \alpha \cong \sigma \oplus \theta$*, then*

$$\mathcal{E} \cong E(\beta \circ \alpha) \cong E(\sigma \oplus \theta) \cong E(\sigma) \oplus E(\theta).$$

We now prove a criterion for the indecomposability of an extension $\mathcal{E} \in {}^p\mathbf{Perv}(X)$ of a fixed object $\mathcal{F} \in {}^p\mathbf{Perv}(U)$.

Theorem 5.1.4.8. *Let* X *be a topologically stratified space with finitely many strata, all with finite fundamental group. Let* $S \subset X$ *be a closed stratum such that* $|\pi_1 S| < \infty$*. Then, an object* $\mathcal{E} \in {}^p\mathbf{Perv}(X)$ *is indecomposable if and only if* $\mathcal{E}$ *is small and the morphism* $\beta \circ \alpha : P_! \mathcal{E} \to P_* \mathcal{E}$ *of (5.2) does not split.*

Proof. Assume $\mathcal{E}$ is small and $\beta \circ \alpha$ does not split. Suppose $\mathcal{E}$ decomposable, for instance

$\mathcal{E} \cong \mathcal{C} \oplus \mathcal{D}$. If $\mathcal{C}$ is supported on S, that is $\mathcal{C} \cong i_* i^* \mathcal{C}$ then

$$\begin{aligned} 0 &\cong {}^p i^{!*}\mathcal{E} \cong {}^p i^{!*}\mathcal{C} \oplus {}^p i^{!*}\mathcal{D} \\ &\cong \mathcal{C} \oplus {}^p i^{!*}\mathcal{D} \end{aligned}$$

so $\mathcal{C} \cong 0$. Similarly, if $\mathcal{D}$ is supported on S one can conclude that $\mathcal{D} \cong 0$. Assume then that neither $\mathcal{C}$ nor $\mathcal{D}$ are supported on S. We can consider the following diagram

$$\begin{tikzcd} P_! \mathcal{E} \arrow[r, "\beta \circ \alpha"] \arrow[d, "\cong"] & P_* \mathcal{E} \arrow[d, "\cong"] \\ P_! \mathcal{C} \oplus P_! \mathcal{D} \arrow[r] & P_* \mathcal{C} \oplus P_* \mathcal{D} \end{tikzcd}$$

Since $j^*\mathcal{C}, j^*\mathcal{D} \ncong 0$, then ${}^p j_{!*} j^* \mathcal{C}, {}^p j_{!*} j^* \mathcal{D} \ncong 0$ and $P_! \mathcal{C}, P_! \mathcal{D}, P_* \mathcal{C}, P_* \mathcal{D} \ncong 0$. By the functoriality of the projection functors $P_!$ and P_*, we have that $\beta \circ \alpha$ has a block diagonal matrix. Hence $\beta \circ \alpha$ splits as a direct sum, which is a contradiction.

Now suppose $\mathcal{E}$ is indecomposable. Since $|\pi_1(S)| < \infty$ for any stratum $S \subset X$, the category ${}^p\mathbf{Perv}(S)$ is semi-simple. Then, there exist morphisms $\delta' \in \mathrm{Hom}_{{}^p\mathbf{Perv}(S)}({}^p i^{!*}\mathcal{E}, {}^p i^! \mathcal{E})$ and $\gamma' \in \mathrm{Hom}_{{}^p\mathbf{Perv}(S)}({}^p i^{*}\mathcal{E}, {}^p i^{!*} \mathcal{E})$ which make the diagram

$$\begin{tikzcd} & {}^p i^{!*}\mathcal{E} \arrow[dl, dashed, "\delta'"'] \arrow[dr, "\gamma"] & \\ {}^p i^! \mathcal{E} \arrow[ur, hook, "\delta"] \arrow[rr, "\epsilon"] \arrow[dr, hook, "i"'] & & {}^p i^* \mathcal{E} \arrow[ul, dashed, two heads, "\gamma'"'] \\ & \mathcal{E} \arrow[ur, two heads, "p"'] & \end{tikzcd}$$

commute and such that

$$\begin{aligned} \gamma' \circ p \circ i \circ \delta' &= \gamma' \circ \gamma \circ \delta \circ \delta' \\ &= \delta' \circ \delta = \mathrm{id}_{{}^p i^{!*}\mathcal{E}} \\ &= \gamma' \circ \gamma = \mathrm{id}_{{}^p i^{!*}\mathcal{E}}. \end{aligned}$$

Let us consider the morphism $\bar{\delta} = i \circ \delta'$ and $\bar{\gamma} = \gamma' \circ p$ which are a monomorphism and an epimorphism respectively. Then, we have

$$ {}^p i^{!*}\mathcal{E} \underset{\bar{\delta}}{\overset{\bar{\gamma}}{\twoheadleftarrow\!\!\longrightarrow}} \mathcal{E} $$

with

$$ \bar{\gamma} \circ \bar{\delta} = \mathrm{id}_{{}^p i^{!*}\mathcal{E}}, $$

that is

$$ \mathcal{E} \cong {}^p i^{!*}\mathcal{E} \oplus \mathcal{E}' $$

for some $\mathcal{E}'$. Hence ${}^p i^{!*}\mathcal{E}$ is a summand of $\mathcal{E}$ supported on S. Since $\mathcal{E}$ is indecomposable by hypothesis we have

$$ {}^p i^{!*}\mathcal{E} = 0 \quad \text{i.e. } \mathcal{E} \text{ is small.} $$

Moreover, if $\beta \circ \alpha : P_! \mathcal{E} \to P_* \mathcal{E}$ decomposes as

$$ \begin{tikzcd} P_!\mathcal{E} \arrow[r, "\beta\circ\alpha"] \arrow[d, "\cong"'] & P_*\mathcal{E} \arrow[d, "\cong"] \\ \mathcal{A} \oplus \mathcal{A}' \arrow[r, "\sigma\oplus\sigma'"'] & \mathcal{B} \oplus \mathcal{B}' \end{tikzcd} $$

by Remark 5.1.4.7 we have either $E(\sigma) \cong 0$ or $E(\sigma') \cong 0$. This implies either $\mathcal{A} \cong \mathcal{B} \cong 0$ or $\mathcal{A}' \cong \mathcal{B}' \cong 0$. Hence $\beta \circ \alpha : P_! \mathcal{E} \to P_* \mathcal{E}$ does not split. $\square$

Remark 5.1.4.9. *Note that we do not require $\mathcal{F} \in {}^p\mathbf{Perv}(U)$ to be indecomposable. If it is so, then $\beta \circ \alpha$ does not split. Hence, in Theorem 5.1.4.8 one has only to check the smallness hypothesis. However, if $\beta \circ \alpha$ does not split, it is not true that $\mathcal{F} \in {}^p\mathbf{Perv}(U)$ has to be indecomposable.*

5.2 Nearby Perversities

In this section we introduce nearby perversities, that is perversities obtained by modifying a given perversity by one on a closed union of strata. We analyse how simple, projective and injective objects change under such modifications. This approach, combined with the results of Section 5.1.4, gives a better understanding of perverse sheaves relative to different perversities.

Let X be a topologically stratified space with finitely many strata, each with finite fundamental group and $\Bbbk$ an algebraically closed field with characteristic not dividing the order of the fundamental groups of the strata. Let $j : U \hookrightarrow X$ and $i : Z \hookrightarrow X$ denote complementary inclusions of open and closed unions of strata respectively. Let p be a perversity on X. We define two new perversities by varying the value of p by one on Z as follows:

$$q(S) = \begin{cases} p(S) & \text{if } S \subset U \\ p(S) - 1 & \text{if } S \subset Z \end{cases} \quad \text{and} \quad r(S) = \begin{cases} p(S) & \text{if } S \subset U \\ p(S) + 1 & \text{if } S \subset Z \end{cases}. \tag{5.7}$$

Example 5.2.0.1. *Let $X = \mathbb{P}^1$ stratified as in Example 2.3.1.6.iii) and denote a perversity p on it by*

$$p(S) = (p(S_1), p(S_0));$$

in particular, the middle perversity is

$$m(S) = (-1, 0).$$

The two nearby perversities defined in (5.7) are

$$q(S) = (-1, -1)$$
$$r(S) = (-1, +1)$$

which can be shifted by adding one to q and subtracting one to r to get the more familiar

$$q(S) = (0, 0)$$
$$r(S) = (-2, 0).$$

Therefore, $q = o$ and $r = t$, that is q and r are the zero and top perversity respectively up to a shift.

5.2.1 Understanding All Perverse Sheaves by Tilting

In this section we explain how starting from a category of perverse sheave one can study other categories of perverse sheaves relative to different perversities by tilting at a torsion pair. Theorem 5.1.4.8 implies that understanding the objects in the edges of the 'ice cream cone' is equivalent to knowing all the indecomposable objects. In turn, this can be used to understand indecomposable objects for nearby perversities. In fact, tilting at a certain torsion pair is equivalent to varying the perversity to a nearby one. Equivalently, ${}^{q}\mathbf{Perv}(X)$ and ${}^{r}\mathbf{Perv}(X)$ are obtained from ${}^{p}\mathbf{Perv}(X)$ by tilting.

Let X be a topologically stratified space with finitely many strata S, each with finite fundamental group, $\Bbbk$ an algebraically closed field such that its characteristic does not divide the order of $\pi_1(S)$ for any stratum $S \subset X$ and p a perversity on X. Let $j : U \hookrightarrow X$ and $i : Z \hookrightarrow X$ be complementary inclusions of open and closed union of strata respectively. Consider ${}^{p}\mathbf{Perv}(X)$ and let q and r be the nearby perversity as defined in (5.7). We denote by

$$\ker {}^{p}i^{*} = \{\mathcal{A} \in {}^{p}\mathbf{Perv}(X) \mid \mathcal{A} \cong P_{!}(\mathcal{F}) \text{ for some } \mathcal{F} \in {}^{p}\mathbf{Perv}(X)\}$$

the subcategory having as objects $\mathcal{A} \in {}^{p}\mathbf{Perv}(X)$ which are extensions of perverse sheaves in ${}^{p}\mathbf{Perv}(U)$ and such that any $\mathcal{A}$ has no quotient on Z.

Lemma 5.2.1.1. *The pair* $(\ker {}^{p}i^{*}, i_{*}{}^{p}\mathbf{Perv}(Z))$ *is a torsion pair in* ${}^{p}\mathbf{Perv}(X)$.

Proof. We need to show, see Definition 2.1.5.6, that:

i) $\mathrm{Hom}_{{}^{p}\mathbf{Perv}(X)}(\mathcal{A}, \mathcal{B}) = 0$ for any $\mathcal{A} \in \ker {}^{p}i^{*}$ and $\mathcal{B} \in i_{*}{}^{p}\mathbf{Perv}(Z)$. This follows from adjunction, see Section 2.3.5.

ii) For any $\mathcal{E} \in {}^{p}\mathbf{Perv}(X)$ there is a unique short exact sequence (up to isomorphism) of the form

$$0 \to \mathcal{A} \to \mathcal{E} \to \mathcal{B} \to 0$$

where $\mathcal{A} \in \ker {}^{p}i^{*}$, that is $\mathcal{A} \cong P_{!}(\mathcal{F})$ for some $\mathcal{F} \in {}^{p}\mathbf{Perv}(X)$, and $\mathcal{B} \cong i_{*}{}^{p}i^{*}(\mathcal{E}) \in i_{*}{}^{p}\mathbf{Perv}(Z)$. This follows from Lemma 5.1.2.3.

$\square$

Definition 5.2.1.2. *A non-empty subcategory $\mathcal{C}$ of an abelian category $\mathcal{A}$ is a* **Serre subcategory** *if for every short exact sequence $0 \to A' \to A \to A'' \to 0$ in $\mathcal{A}$ we have $A \in \mathcal{C}$ if and only if $A', A'' \in \mathcal{C}$.*

Remark 5.2.1.3. *Definition 5.2.1.2 is equivalent to ask that $\mathcal{C}$ is closed under sub-objects, quotients and extensions.*

Remark 5.2.1.4. *Since $i_{*}{}^{p}\mathbf{Perv}(Z)$ is a Serre subcategory of ${}^{p}\mathbf{Perv}(X)$, it is a torsion-free class, see Remark 2.1.5.7 (and also a torsion class).*

We now show that for nearby perversities q and r we can express the categories ${}^q\mathbf{Perv}(X)$ and ${}^r\mathbf{Perv}(X)$ as forward and backward HRS-tilt, see Definition 2.1.5.8, of ${}^p\mathbf{Perv}(X)$ at some specific torsion pair.

Lemma 5.2.1.5. *Let q and r be the nearby perversities defined in (5.7). Then, ${}^q\mathbf{Perv}(X)$ and ${}^r\mathbf{Perv}(X)$ are the forward and backward HRS-tilt of ${}^p\mathbf{Perv}(X)$ at the torsion pair $(\ker {}^p i^*, i_* {}^p\mathbf{Perv}(Z))$.*

Proof. We prove the statement for the nearby perversity q as the other case is dual. We have ${}^q\mathbf{Perv}(X) \subset \langle {}^p\mathbf{Perv}(X), {}^p\mathbf{Perv}(X)[1] \rangle$ and so ${}^q\mathbf{Perv}(X)$ is a tilt of ${}^p\mathbf{Perv}(X)$ at a torsion theory, more specifically is the forward tilt at $(\mathscr{T}, \mathscr{F})$, where

$$\begin{aligned} \mathscr{T} &\cong {}^p\mathbf{Perv}(X) \cap {}^q\mathbf{Perv}(X) \\ \mathscr{F} &\cong \mathscr{T}^\perp \cong \{\mathcal{F} \in {}^p\mathbf{Perv}(X) \mid \mathrm{Hom}(\mathcal{E}, \mathcal{F}) = 0 \ \ \forall \mathcal{E} \in \mathscr{T}\}. \end{aligned}$$

By definition, we have that $\mathcal{E} \in {}^p\mathbf{Perv}(X) \cap {}^q\mathbf{Perv}(X)$ is equivalent to

i) $j^*\mathcal{E} \in {}^p\mathbf{Perv}(U) \cong {}^q\mathbf{Perv}(U)$.

ii) $i^*\mathcal{E} \in {}^p\mathrm{D}^{\leq 0}(Z) \cap {}^q\mathrm{D}^{\leq 0}(Z) = {}^p\mathrm{D}^{\leq 0}(Z) \cap {}^p\mathrm{D}^{\leq -1}(Z) = {}^p\mathrm{D}^{\leq -1}(Z)$.

iii) $i^!\mathcal{E} \in {}^p\mathrm{D}^{\geq 0}(Z) \cap {}^q\mathrm{D}^{\geq 0}(Z) = {}^p\mathrm{D}^{\geq 0}(Z) \cap {}^p\mathrm{D}^{\geq -1}(Z) = {}^p\mathrm{D}^{\geq 0}(Z)$.

Equivalently, $\mathcal{E} \in {}^p\mathbf{Perv}(X) \cap {}^q\mathbf{Perv}(X)$ if and only if $\mathcal{E} \in {}^p\mathbf{Perv}(X)$ and ${}^p i^*\mathcal{E} \cong 0$, that is $\mathcal{E} \in \ker({}^p i^* : {}^p\mathbf{Perv}(X) \to {}^p\mathbf{Perv}(Z))$. This is also equivalent to $\mathcal{E} \in {}^p\mathbf{Perv}(X)$ being of the form $P_!\mathcal{E}'$ for some $\mathcal{E}' \in {}^p\mathbf{Perv}(X)$, see Lemma 5.1.2.3. By adjunction, if $\mathcal{F} \in \mathrm{im}(i_* : {}^p\mathbf{Perv}(Z) \to {}^p\mathbf{Perv}(X)) \cong {}^p\mathbf{Perv}(Z)$, then

$$\mathrm{Hom}(\mathcal{E}, i_*\mathcal{F}) \cong \mathrm{Hom}({}^p i^*\mathcal{E}, \mathcal{F}) \cong 0$$

for $\mathcal{E} \in {}^p\mathbf{Perv}(X) \cap {}^q\mathbf{Perv}(X)$. Hence, $\mathrm{im} i_* \subset ({}^p\mathbf{Perv}(X) \cap {}^q\mathbf{Perv}(X))^\perp$. Since there is a short exact sequence in ${}^p\mathbf{Perv}(X)$

$$0 \to P_!\mathcal{G} \to \mathcal{G} \to i_* {}^p i^*\mathcal{G} \to 0$$

for any $\mathcal{G} \in {}^p\mathbf{Perv}(X)$, we have that $\mathrm{im} i_* = ({}^p\mathbf{Perv}(X) \cap {}^q\mathbf{Perv}(X))^\perp$. Hence ${}^q\mathbf{Perv}(X)$ is the forward tilt of ${}^p\mathbf{Perv}(X)$ at the torsion theory $(\ker {}^p i^*, \mathrm{im} i_*) = (\ker {}^p i^*, {}^p\mathbf{Perv}(Z))$. $\square$

Remark 5.2.1.6. *One can check the conditions of Theorem 2.1.5.11 to establish if* ${}^{q}\mathbf{Perv}(X)$ *and* ${}^{r}\mathbf{Perv}(X)$ *are faithful or not.*

Example 5.2.1.7. *Let us consider* $X = \mathbb{P}^1$ *stratified as in Example 2.3.1.6.iii) and let* $p = m$ *the middle perversity. We claim that the forward HRS-tilt at the torsion pair* $(\mathscr{T}, \mathscr{F}) = (\ker {}^{p}i^{*}, i_{*}{}^{p}\mathbf{Perv}(Z))$, *which is* $\mathcal{B} \simeq {}^{q}\mathbf{Perv}(X)$ *as noted above, is not faithful. The heart* ${}^{p}\mathbf{Perv}(X)$ *is faithful, see [Bei87b] and*

$$\mathscr{T} \cong \langle {}^{p}j_{!}\Bbbk_U[1], {}^{p}j_{!*}\Bbbk_U[1]\rangle \cong \langle j_{!}\Bbbk_U[1], \Bbbk_X[1]\rangle$$
$$\mathscr{F} \cong \langle \Bbbk_Z\rangle$$

We want to show that the object

$${}^{p}\mathcal{I}_U \cong {}^{p}j_{*}\Bbbk_U[1] \cong j_{*}\Bbbk_U[1]$$

does not fit into a short exact sequence of the form

$$0 \to F^0 \to F^1 \to {}^{p}\mathcal{I}_U \to T^0 \to T^1 \to 0$$

where $F^0, F^1 \in \mathscr{F}$ *and* $T^0, T^1 \in \mathscr{T}$. *This follows from the fact that* ${}^{p}\mathcal{I}_U \notin \mathscr{T}$ *and* ${}^{p}\mathcal{I}_U \notin \mathscr{F}$; *moreover* $\mathrm{Hom}(F, {}^{p}\mathcal{I}_U) = 0$ *for any* $F \in \mathscr{F}$ *and there is no monomorphism in* $\mathrm{Hom}({}^{p}\mathcal{I}_U, T)$ *for* $T \in \mathscr{T}$ *(and for any indecomposable* T *in* ${}^{p}\mathbf{Perv}(X)$ *in general). Therefore, Theorem 2.1.5.11 guarantees that* ${}^{q}\mathbf{Perv}(X)$ *is not faithful. Dually, the backward HRS-tilt at* ${}^{p}\mathbf{Perv}(X)$ *is also not faithful. In conclusion, for* $X = \mathbb{P}^1$ *with the affine stratification the only faithful perverse heart in* $\mathbf{D}_c(X)$ *is given by perverse sheaves for the middle perversity.*

The following result is another way to check if ${}^{p}\mathbf{Perv}(X)$ is not faithful.

Lemma 5.2.1.8. *Let* X *be an* n*-dimensional topologically stratified space, where the strata* $S_i \subset X$ *are contractible for* $i = 0, \dots, n$. *Let* $\mathcal{E} \in {}^{p}\mathbf{Perv}(X)$ *be either simple projective or simple injective supported on a stratum* S_i *such that* $H^k(S_n) \neq 0$ *for some* $k > 0$. *Then* ${}^{p}\mathbf{Perv}(X)$ *is not faithful.*

Proof. We consider the case when $\mathcal{S}_n \in {}^{p}\mathbf{Perv}(X)$ is simple and projective as the other is completely dual. For any $\mathcal{E} \in {}^{p}\mathbf{Perv}(X)$ we have

$$\mathrm{Ext}^i_{{}^{p}\mathbf{Perv}(X)}(\mathcal{S}_n, \mathcal{E}) \cong \mathrm{Ext}^i_{{}^{p}\mathbf{Perv}(X)}(\mathcal{P}_n, \mathcal{E}) \cong 0.$$

On the other hand

$$\mathrm{Ext}^i_{\mathbf{D}_c(X)}(\mathcal{S}_n, \mathcal{S}_n) \cong H^i(S_n) \neq 0.$$

Therefore ${}^p\mathbf{Perv}(X)$ cannot be a faithful heart. □

5.2.2 How Simple Objects Change

In this section we want to study how simple objects change when moving to a nearby perversity. Let $\mathcal{L} \in \mathbf{Loc}(S)$ for some stratum $S \subset X$ and ${}^p\mathcal{S}_{\mathcal{L}} \in {}^p\mathbf{Perv}(X)$ be the corresponding simple p-perverse sheaf. We want to understand ${}^q\mathcal{S}_{\mathcal{L}} \in {}^q\mathbf{Perv}(X)$ and ${}^r\mathcal{S}_{\mathcal{L}} \in {}^r\mathbf{Perv}(X)$.

Lemma 5.2.2.1. *Let $\mathcal{E} \in {}^p\mathbf{Perv}(U)$ and $\mathcal{F} \in {}^p\mathbf{Perv}(Z)$, then $\mathrm{Ext}^1({}^pj_!\mathcal{E}, i_*\mathcal{F}) = 0$.*

Proof. Let us consider a non zero element in $\mathrm{Ext}^1({}^pj_!\mathcal{E}, i_*\mathcal{F})$, that is a non-split short exact sequence of the form

$$0 \to i_*\mathcal{F} \to \mathcal{G} \to {}^pj_!\mathcal{E} \to 0. \tag{5.8}$$

In addition, we have $j^*\mathcal{G} \cong \mathcal{E}$, since j^* is exact and $j^*i_* = 0$, and an epimorphism $\mathcal{F} \twoheadrightarrow {}^pi^*\mathcal{G}$, as ${}^pi^*$ is left adjoint to i_* and ${}^pi^*{}^pj_! = 0$, see Section 2.3.5. Furthermore, it is enough to show the claim for $\mathcal{F}$ a simple object. Since we are assuming $\mathcal{F} \in {}^p\mathbf{Perv}(Z)$ simple, it cannot have non-trivial quotients and sub-objects supported on Z, therefore either ${}^pi^*\mathcal{G} \cong \mathcal{F}$ or ${}^pi^*\mathcal{G} \cong 0$. The first case implies that (5.8) splits. The second case cannot happen as it would imply ${}^pj_!\mathcal{E} \cong {}^pj_!j^*\mathcal{G} \twoheadrightarrow \mathcal{G}$ and then, in terms of length, see Definition 2.1.1.12,

$$\ell(\mathcal{G}) \leq \ell({}^pj_!j^*\mathcal{G}) \;\Rightarrow\; \ell({}^pj_!\mathcal{E}) + \ell(i_*\mathcal{F}) \leq \ell({}^pj_!\mathcal{E}).$$

This concludes the proof. □

Remark 5.2.2.2. *Lemma 5.2.2.1 can be interpreted by saying that the object ${}^pj_!\mathcal{E}$ is quite close to being projective.*

Proposition 5.2.2.3. *The simple objects in ${}^q\mathbf{Perv}(X)$ are of the form*

$$ {}^q\mathcal{S}_{\mathcal{L}} = \begin{cases} {}^pj_!j^*{}^p\mathcal{S}_{\mathcal{L}} & \mathcal{L} \in \mathbf{Loc}(S),\ S \subset U \\ {}^p\mathcal{S}_{\mathcal{L}}[1] & \mathcal{L} \in \mathbf{Loc}(S),\ S \subset Z \end{cases}$$

Dually, the simple objects in ${}^{r}\mathbf{Perv}(X)$ *are*

$$ {}^{r}\mathcal{S}_{\mathcal{L}} = \begin{cases} {}^{p}j_{*}j^{*}{}^{p}\mathcal{S}_{\mathcal{L}} & \mathcal{L} \in \mathbf{Loc}(S),\ S \subset U \\ {}^{p}\mathcal{S}_{\mathcal{L}}[-1] & \mathcal{L} \in \mathbf{Loc}(S),\ S \subset Z \end{cases} $$

Proof. We prove only the first characterisation of simple objects, as the two cases are dual to each other. If $S \subset Z$ the statement is clear as the new simple object ${}^{q}\mathcal{S}_{\mathcal{L}}$ is just a shift of ${}^{p}\mathcal{S}_{\mathcal{L}}$. Let us suppose $\mathcal{L} \in \mathbf{Loc}(S)$ where $S \subset U$. In order to prove the claim it is enough to show that ${}^{p}j_{!}j^{*}{}^{p}\mathcal{S}_{\mathcal{L}}$ extends $\mathcal{L}[-p(S)] \cong \mathcal{L}[-q(S)]$ and it has neither quotients nor sub-objects in ${}^{q}\mathbf{Perv}(\overline{S} \smallsetminus S)$, see Lemma 2.3.5.6. We then have two cases

1. If $\mathcal{M} \in \mathbf{Loc}(T)$ where $T \subset Z$, then ${}^{q}\mathcal{S}_{\mathcal{M}} \cong {}^{p}\mathcal{S}_{\mathcal{M}}[1]$. Hence

$$ \begin{aligned} \mathrm{Hom}({}^{q}\mathcal{S}_{\mathcal{M}}, {}^{p}j_{!}j^{*}{}^{p}\mathcal{S}_{\mathcal{L}}) &\cong \mathrm{Hom}({}^{p}\mathcal{S}_{\mathcal{M}}[1], {}^{p}j_{!}j^{*}{}^{p}\mathcal{S}_{\mathcal{L}}) \\ &\cong \mathrm{Ext}^{-1}({}^{p}\mathcal{S}_{\mathcal{M}}, {}^{p}j_{!}j^{*}{}^{p}\mathcal{S}_{\mathcal{L}}) = 0, \end{aligned} $$

therefore ${}^{p}j_{!}j^{*}{}^{p}\mathcal{S}_{\mathcal{L}}$ has no sub-object supported on Z. Moreover

$$ \begin{aligned} \mathrm{Hom}({}^{p}j_{!}j^{*}{}^{p}\mathcal{S}_{\mathcal{L}}, {}^{q}\mathcal{S}_{\mathcal{M}}) &\cong \mathrm{Hom}({}^{p}j_{!}j^{*}{}^{p}\mathcal{S}_{\mathcal{L}}, {}^{p}\mathcal{S}_{\mathcal{M}}[1]) \\ &\cong \mathrm{Ext}^{1}({}^{p}j_{!}j^{*}{}^{p}\mathcal{S}_{\mathcal{L}}, {}^{p}\mathcal{S}_{\mathcal{M}}) \cong 0 \quad \text{by Lemma 5.2.2.1.} \end{aligned} $$

That is ${}^{p}j_{!}j^{*}{}^{p}\mathcal{S}_{\mathcal{L}}$ has no quotient supported on Z.

2. If $\mathcal{M} \in \mathbf{Loc}(T)$ where $T \subset U \cap (\overline{S} \smallsetminus S)$, then we know that ${}^{q}\mathcal{S}_{\mathcal{M}} \in {}^{p}\mathbf{Perv}(X)$. Since $j^{*}{}^{q}\mathcal{S}_{\mathcal{M}}$ is supported on $U \cap (\overline{S} \smallsetminus S)$, we have

$$ \mathrm{Hom}({}^{p}j_{!}j^{*}{}^{p}\mathcal{S}_{\mathcal{L}}, {}^{q}\mathcal{S}_{\mathcal{M}}) \cong \mathrm{Hom}(j^{*}{}^{p}\mathcal{S}_{\mathcal{L}}, j^{*}{}^{q}\mathcal{S}_{\mathcal{M}}) \cong 0, $$

because $j^{*}{}^{p}\mathcal{S}_{\mathcal{L}}$ is simple in ${}^{p}\mathbf{Perv}(U \cap \overline{S})$ and therefore has no quotient supported in ${}^{p}\mathbf{Perv}(U \cap (\overline{S} \smallsetminus S))$. It remains to show that ${}^{p}j_{!}j^{*}{}^{p}\mathcal{S}_{\mathcal{L}}$ has no sub-object supported on $\overline{S} \smallsetminus S$. Let us consider the triangle

$$ j_{!}j^{*}{}^{q}\mathcal{S}_{\mathcal{M}} \to {}^{q}\mathcal{S}_{\mathcal{M}} \to i_{*}i^{*}{}^{q}\mathcal{S}_{\mathcal{M}} \to j_{!}j^{*}{}^{q}\mathcal{S}_{\mathcal{M}}[1] $$

and apply the functor $\mathrm{Hom}(-, {}^p j_! j^{*p}\mathcal{S}_{\mathcal{L}})$ to get the long exact sequence

$$\ldots \to \mathrm{Hom}(i_* i^{*q}\mathcal{S}_{\mathcal{M}}, {}^p j_! j^{*p}\mathcal{S}_{\mathcal{L}}) \to \mathrm{Hom}({}^q\mathcal{S}_{\mathcal{M}}, {}^p j_! j^{*p}\mathcal{S}_{\mathcal{L}}) \to \mathrm{Hom}(j_! j^{*q}\mathcal{S}_{\mathcal{M}}, {}^p j_! j^{*p}\mathcal{S}_{\mathcal{L}}) \to \ldots \tag{5.9}$$

For the first term in the above long exact sequence we have

$$\mathrm{Hom}(i_* i^{*q}\mathcal{S}_{\mathcal{M}}, {}^p j_! j^{*p}\mathcal{S}_{\mathcal{L}}) \cong \mathrm{Hom}(i^{*q}\mathcal{S}_{\mathcal{M}}, i^{!p} j_! j^{*p}\mathcal{S}_{\mathcal{L}}) \cong 0$$

since

$$\begin{aligned} i^{*q}\mathcal{S}_{\mathcal{M}} &\in {}^q\mathrm{D}^{<0}(Z) = {}^q\mathrm{D}^{\leq -1}(Z) \\ i^{!p} j_! j^{*p}\mathcal{S}_{\mathcal{L}} &\in {}^p\mathrm{D}^{\geq 0}(Z) = {}^q\mathrm{D}^{\geq 1}(Z). \end{aligned}$$

On the other hand, for the last term in (5.9) we have

$$\begin{aligned} \mathrm{Hom}(j_! j^{*q}\mathcal{S}_{\mathcal{M}}, {}^p j_! j^{*p}\mathcal{S}_{\mathcal{L}}) &\cong \mathrm{Hom}(j^{*q}\mathcal{S}_{\mathcal{M}}, j^{*p}\mathcal{S}_{\mathcal{L}}) \\ &\cong \mathrm{Hom}(j^{*p}\mathcal{S}_{\mathcal{M}}, j^{*p}\mathcal{S}_{\mathcal{L}}) \cong 0 \end{aligned}$$

as $j^{*p}\mathcal{S}_{\mathcal{M}}$ and $j^{*p}\mathcal{S}_{\mathcal{L}}$ are distinct simple objects in ${}^p\mathbf{Perv}(U)$. This finishes the proof since the above implies that ${}^p j_! j^{*p}\mathcal{S}_{\mathcal{L}}$ has no sub-object supported on $\overline{S} \smallsetminus S$.

$\square$

What is described in Proposition 5.2.2.3 is an instance of simple-minded mutation, see [KY12].

Example 5.2.2.4. *Let $X = \mathbb{P}^1$ stratified as in Example 2.3.1.6.iii) and consider the middle perversity $p = m$. The two simple objects in ${}^m\mathbf{Perv}(X)$ are*

$$\begin{aligned} {}^m\mathcal{S}_U &\cong \Bbbk_X[1] \\ {}^m\mathcal{S}_Z &\cong i_*\Bbbk_Z. \end{aligned}$$

By Proposition 5.2.2.3 and Example 5.2.0.1, the new simple objects in ${}^q\mathbf{Perv}(X) \cong {}^o\mathbf{Perv}(X)$ are

$$\begin{aligned} {}^o\mathcal{S}_U &\cong {}^p j_! j^{*m}\mathcal{S}_U \cong j_! j^* \Bbbk_X[1] \\ {}^o\mathcal{S}_Z &\cong i_*\Bbbk_Z[1] \end{aligned}$$

which, shifting by -1, become

$$
\begin{aligned}
{}^{o}\mathcal{S}_U &\cong j_!\mathbb{k}_U \\
{}^{o}\mathcal{S}_Z &\cong i_*\mathbb{k}_Z.
\end{aligned}
$$

Dually, the new simple objets in ${}^{r}\mathbf{Perv}(X) \cong {}^{t}\mathbf{Perv}(X)$ *are*

$$
\begin{aligned}
{}^{t}\mathcal{S}_U &\cong {}^{p}j_*j^{*m}\mathcal{S}_U \cong j_*j^*\mathbb{k}_X[1] \\
{}^{t}\mathcal{S}_Z &\cong i_*\mathbb{k}_Z[-1]
\end{aligned}
$$

which, by shifting by +1, become

$$
\begin{aligned}
{}^{t}\mathcal{S}_U &\cong j_*j^*\mathbb{k}_X[2] \\
{}^{t}\mathcal{S}_Z &\cong i_*\mathbb{k}_Z.
\end{aligned}
$$

This agrees with Example 2.3.7.4.

5.2.3 How Projective Covers and Injective Hulls Change

Let ${}^{p}\mathcal{P}_{\mathcal{L}} \in {}^{p}\mathbf{Perv}(X)$ denote the projective cover of a simple object ${}^{p}\mathcal{S}_{\mathcal{L}}$ for a local system $\mathcal{L} \in \mathbf{Loc}(S)$, where $S \subset X$ is some stratum. We want to understand how projective covers and injective hulls change for nearby perversities. Although we do not know how to do this in general, we can characterise how some indecomposable projective and injective object change under some extra assumptions on ${}^{p}\mathbf{Perv}(X)$.

Proposition 5.2.3.1. *Let* ${}^{p}\mathbf{Perv}(X)$ *be a faithful heart and recall that, by Lemma 3.3.3.2,* $j^{*p}\mathcal{P}_{\mathcal{L}}$ *is the projective cover of* ${}^{p}\mathcal{S}_{\mathcal{L}}$ *in* ${}^{p}\mathbf{Perv}(U)$ *for* $\mathcal{L} \in \mathbf{Loc}(S)$ *where* $S \subset U$*. Then* ${}^{q}\mathcal{P}_{\mathcal{L}} \cong {}^{p}j_!j^{*p}\mathcal{P}_{\mathcal{L}}$ *is the projective cover of* ${}^{q}\mathcal{S}_{\mathcal{L}} \cong {}^{p}j_!j^{*p}\mathcal{S}_{\mathcal{L}}$ *in* ${}^{q}\mathbf{Perv}(X)$*.*

Proof. In order to prove that ${}^{q}\mathcal{P}_{\mathcal{L}} \cong {}^{p}j_!j^{*p}\mathcal{P}_{\mathcal{L}}$ is the projective cover of ${}^{q}\mathcal{S}_{\mathcal{L}} \cong {}^{p}j_!j^{*p}\mathcal{S}_{\mathcal{L}}$ in ${}^{q}\mathbf{Perv}(X)$, we check that the hypothesis of Lemma 2.1.2.6 hold.

i) There exists an epimorphism ${}^{q}\mathcal{P}_{\mathcal{L}} \twoheadrightarrow {}^{q}\mathcal{S}_{\mathcal{L}}$. This follows from the fact that ${}^{p}j_!$ is left adjoint to the exact functor j^*, see (2.6).

ii) For any simple object ${}^q\mathcal{S}_{\mathcal{M}} \in {}^q\mathbf{Perv}(X)$ we need to show that

$$\mathrm{Hom}_{{}^q\mathbf{Perv}(X)}({}^q\mathcal{P}_{\mathcal{L}}, {}^q\mathcal{S}_{\mathcal{M}}) \cong \begin{cases} \mathbb{k} & \text{if } \mathcal{M} = \mathcal{L} \\ 0 & \text{otherwise.} \end{cases}$$

There are two cases:

<u>Case 1:</u> Let $\mathcal{M} \in \mathbf{Loc}(T)$ for $T \subset Z$. Then

$$\begin{aligned} \mathrm{Hom}_{{}^q\mathbf{Perv}(X)}({}^q\mathcal{P}_{\mathcal{L}}, {}^q\mathcal{S}_{\mathcal{M}}) &\cong \mathrm{Hom}_{{}^p\mathbf{Perv}(X)}({}^p j_! j^{*p}\mathcal{P}_{\mathcal{L}}, {}^p\mathcal{S}_{\mathcal{M}}[1]) \\ &\cong \mathrm{Ext}^1({}^p j_! j^{*p}\mathcal{P}_{\mathcal{L}}, {}^p\mathcal{S}_{\mathcal{M}}) \cong 0 \end{aligned}$$

since ${}^p j_! j^{*p}\mathcal{P}_{\mathcal{L}}$ is projective in ${}^p\mathbf{Perv}(X)$ by Lemma 3.3.3.1.

<u>Case 2:</u> Let $\mathcal{M} \in \mathbf{Loc}(T)$ where $T \subset U$. Then

$$\begin{aligned} \mathrm{Hom}_{{}^q\mathbf{Perv}(X)}({}^q\mathcal{P}_{\mathcal{L}}, {}^q\mathcal{S}_{\mathcal{M}}) &\cong \mathrm{Hom}_{{}^p\mathbf{Perv}(X)}({}^p j_! j^{*p}\mathcal{P}_{\mathcal{L}}, {}^p j_! j^{*p}\mathcal{S}_{\mathcal{M}}) \\ &\cong \mathrm{Hom}_{{}^p\mathbf{Perv}(U)}(j^{*p}\mathcal{P}_{\mathcal{L}}, j^{*p}\mathcal{S}_{\mathcal{M}}) \\ &\cong \begin{cases} \mathbb{k} & \text{if } \mathcal{M} = \mathcal{L} \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

since $j^{*p}\mathcal{P}_{\mathcal{L}}$ is the projective cover of the simple object ${}^p\mathcal{S}_{\mathcal{L}}$ in ${}^p\mathbf{Perv}(U)$.

iii) For any simple object ${}^q\mathcal{S}_{\mathcal{M}} \in {}^q\mathbf{Perv}(X)$ we need to prove that $\mathrm{Ext}^1_{{}^q\mathbf{Perv}(X)}({}^q\mathcal{P}_{\mathcal{L}}, {}^q\mathcal{S}_{\mathcal{M}})$ vanishes. Again, there are two cases:

<u>Case 1:</u> Let $\mathcal{M} \in \mathbf{Loc}(S)$ where $S \subset Z$. Then

$$\begin{aligned} \mathrm{Ext}^1_{{}^q\mathbf{Perv}(X)}({}^q\mathcal{P}_{\mathcal{L}}, {}^q\mathcal{S}_{\mathcal{M}}) &\cong \mathrm{Ext}^1_{{}^p\mathbf{Perv}(X)}({}^p j_! j^{*p}\mathcal{P}_{\mathcal{L}}, {}^p\mathcal{S}_{\mathcal{M}}[1]) \\ &\cong \mathrm{Ext}^2_{{}^p\mathbf{Perv}(X)}({}^p j_! j^{*p}\mathcal{P}_{\mathcal{L}}, {}^p\mathcal{S}_{\mathcal{M}}) \cong 0 \end{aligned}$$

since ${}^p\mathbf{Perv}(X)$ is faithful so that $\mathrm{Ext}^2_{\mathbf{D}_c(X)} \cong \mathrm{Ext}^2_{{}^p\mathbf{Perv}(X)}$ and ${}^p j_! j^{*p}\mathcal{P}_{\mathcal{L}}$ is projective in it.

<u>Case 2:</u> Let $\mathcal{M} \in \mathbf{Loc}(S)$ where $S \subset U$. Then

$$\mathrm{Ext}^1_{{}^q\mathbf{Perv}(X)}({}^q\mathcal{P}_{\mathcal{L}}, {}^q\mathcal{S}_{\mathcal{M}}) \cong \mathrm{Ext}^1_{{}^p\mathbf{Perv}(X)}({}^p j_! j^{*p}\mathcal{P}_{\mathcal{L}}, {}^p j_! j^{*p}\mathcal{S}_{\mathcal{M}}) \cong 0$$

as ${}^p j_! j^{*p}\mathcal{P}_{\mathcal{L}}$ is projective in ${}^p\mathbf{Perv}(X)$.

□

Example 5.2.3.2. *Let $X = \mathbb{P}^1$ stratified as in Example 2.3.1.6.iii) and consider the middle perversity $p = m$. We have that ${}^m\mathbf{Perv}(X)$ is faithful, see [Bei87b]. Let q be the nearby perversity defined in (5.7), then, as noted in Example 5.2.0.1, q is the zero perversity. By Proposition 5.2.3.1 the projective cover ${}^q\mathcal{P}_U$ in ${}^q\mathbf{Perv}(X) \cong {}^o\mathbf{Perv}(X)$ coincides with the simple object ${}^o\mathcal{S}_U \cong j_!\Bbbk_U$. That is, the object $j_!\Bbbk_U \in {}^o\mathbf{Perv}(X)$ is simple and projective.*

Proposition 5.2.3.3. *Let $\mathcal{M} \in \mathbf{Loc}(S)$ where $S \subset U$ and recall that $j^{*p}\mathcal{I}_{\mathcal{M}}$ is the injective hull of $j^{*p}\mathcal{S}_{\mathcal{M}}$ in ${}^p\mathbf{Perv}(U)$. Then, ${}^q\mathcal{I}_{\mathcal{M}} \cong {}^p j_{!*} j^{*p}\mathcal{I}_{\mathcal{M}}$ is the injective hull of ${}^q\mathcal{S}_{\mathcal{M}}$ in ${}^q\mathbf{Perv}(X)$.*

Proof. In order to we check that the hypothesis of Lemma 2.1.2.12 hold.

i) There exists a monomorphism ${}^q\mathcal{S}_{\mathcal{M}} \hookrightarrow {}^q\mathcal{I}_{\mathcal{M}}$. This follows from a dual argument to the one used in the proof of Proposition 5.2.3.1.i).

ii) For any simple object ${}^q\mathcal{S}_{\mathcal{L}} \in {}^q\mathbf{Perv}(X)$ we need to prove that

$$\mathrm{Hom}_{{}^q\mathbf{Perv}(X)}({}^q\mathcal{S}_{\mathcal{L}}, {}^q\mathcal{I}_{\mathcal{M}}) \cong \begin{cases} \Bbbk & \text{if } \mathcal{L} = \mathcal{M} \\ 0 & \text{otherwise} \end{cases}.$$

There are two cases:

<u>Case 1:</u> Let $\mathcal{L} \in \mathbf{Loc}(T)$ where $T \subset U$. Then

$$\begin{aligned} \mathrm{Hom}_{{}^q\mathbf{Perv}(X)}({}^q\mathcal{S}_{\mathcal{L}}, {}^q\mathcal{I}_{\mathcal{M}}) &\cong \mathrm{Hom}_{{}^p\mathbf{Perv}(X)}({}^p j_! j^{*p}\mathcal{S}_{\mathcal{L}}, {}^p j_{!*} j^{*p}\mathcal{I}_{\mathcal{M}}) \\ &\cong \mathrm{Hom}_{{}^p\mathbf{Perv}(U)}(j^{*p}\mathcal{S}_{\mathcal{L}}, j^{*p}\mathcal{I}_{\mathcal{M}}) \\ &\cong \begin{cases} \Bbbk & \text{if } \mathcal{L} \cong \mathcal{M} \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

because $j^{*p}\mathcal{S}_{\mathcal{L}}$ and $j^{*p}\mathcal{I}_{\mathcal{M}}$ are respectively simple and indecomposable injective in ${}^p\mathbf{Perv}(U)$.

<u>Case 2:</u> Let $\mathcal{L} \in \mathbf{Loc}(T)$ where $T \subset Z$. Then

$$\mathrm{Hom}_{{}^q\mathbf{Perv}(X)}({}^q\mathcal{S}_{\mathcal{L}}, {}^q\mathcal{I}_{\mathcal{M}}) \cong \mathrm{Hom}_{{}^p\mathbf{Perv}(X)}({}^p\mathcal{S}_{\mathcal{L}}[1], {}^p j_{!*} j^{*p}\mathcal{I}_{\mathcal{M}}) \cong 0.$$

iii) For any simple object ${}^q\mathcal{S}_\mathcal{L} \in {}^q\mathbf{Perv}(X)$ we need to prove that $\mathrm{Ext}^1_{{}^q\mathbf{Perv}(X)}({}^q\mathcal{S}_\mathcal{L}, {}^q\mathcal{I}_\mathcal{M})$ vanishes. Again, there are two cases:

Case 1: Let $\mathcal{L} \in \mathbf{Loc}(T)$ where $T \subset U$. Then

$$\begin{aligned}\mathrm{Ext}^1_{{}^q\mathbf{Perv}(X)}({}^q\mathcal{S}_\mathcal{L}, {}^q\mathcal{I}_\mathcal{M}) &\cong \mathrm{Ext}^1_{{}^p\mathbf{Perv}(X)}({}^p j_! j^{*p}\mathcal{S}_\mathcal{L}, {}^p j_{!*} j^{*p}\mathcal{I}_\mathcal{M}) \\ &\cong \mathrm{Ext}^1_{{}^p\mathbf{Perv}(U)}(j^{*p}\mathcal{S}_\mathcal{L}, j^{*p}\mathcal{I}_\mathcal{M}) \cong 0\end{aligned}$$

since ${}^p\mathcal{I}_\mathcal{M}$ is injective in ${}^p\mathbf{Perv}(U)$.

Case 2: Let $\mathcal{L} \in \mathbf{Loc}(T)$ where $T \subset Z$. Then

$$\begin{aligned}\mathrm{Ext}^1_{{}^q\mathbf{Perv}(X)}({}^q\mathcal{S}_\mathcal{L}, {}^q\mathcal{I}_\mathcal{M}) &\cong \mathrm{Ext}^1_{{}^p\mathbf{Perv}(X)}({}^p\mathcal{S}_\mathcal{L}[1], {}^p j_{!*} j^{*p}\mathcal{I}_\mathcal{M}) \\ &\cong \mathrm{Hom}_{{}^p\mathbf{Perv}(X)}({}^p\mathcal{S}_\mathcal{L}, {}^p j_{!*} j^{*p}\mathcal{I}_\mathcal{M}) \cong 0\end{aligned}$$

as the intermediate extension has no sub-object supported on Z.

$\square$

Remark 5.2.3.4. *Proposition 5.2.3.3 works without the requirement that ${}^p\mathbf{Perv}(X)$ is faithful.*

Example 5.2.3.5. *Let $X = \mathbb{P}^1$ stratified as in Example 2.3.1.6.iii) and consider the middle perversity $p = m$. Let q be the nearby perversity defined in (5.7), then, as noted in Example 5.2.0.1, q is the zero perversity. Since $j^{*m}\mathcal{I}_U \cong j^{*m}\mathcal{S}_U$, by Proposition 5.2.3.3 we have that the injective hull in ${}^q\mathbf{Perv}(X) \cong {}^o\mathbf{Perv}(X)$ of the simple object ${}^o\mathcal{S}_U \cong j_!\mathbb{k}_U$ is (up to a shift)*

$$ {}^o\mathcal{I}_U \cong {}^p j_{!*} j^{*m}\mathcal{S}_U \cong \mathbb{k}_X. $$

There are dual statements to the ones of Proposition 5.2.3.1 and Proposition 5.2.3.3 for the perversity r defined in (5.7).

Proposition 5.2.3.6. *Let $\mathcal{L} \in \mathbf{Loc}(S)$ where $S \subset U$. Recall that $j^{*p}\mathcal{I}_\mathcal{L}$ and $j^{*p}\mathcal{P}_\mathcal{L}$ are the injective hull and projective cover of ${}^p\mathcal{S}_\mathcal{L}$ in ${}^p\mathbf{Perv}(U)$ respectively. Then:*

i) The projective cover of ${}^r\mathcal{S}_\mathcal{L}$ in ${}^r\mathbf{Perv}(X)$ is given by ${}^r\mathcal{P}_\mathcal{L} \cong {}^p j_{!*} j^{*p}\mathcal{P}_\mathcal{L}$.

ii) If ${}^p\mathbf{Perv}(X)$ is faithful, the injective hull of ${}^r\mathcal{S}_\mathcal{L}$ in ${}^r\mathbf{Perv}(X)$ is given by ${}^r\mathcal{I}_\mathcal{L} \cong {}^p\mathcal{I}_\mathcal{L}$.

Proof. The proofs are dual to the ones of Proposition 5.2.3.3 and Proposition 5.2.3.1 respectively. □

Example 5.2.3.7. *Let $X = \mathbb{P}^1$ stratified as in Example 2.3.1.6.iii) and consider the middle perversity $p = m$. Let r be the nearby perversity defined in (5.7), then, as noted in Example 5.2.0.1, r is the top perversity. Since $j^{*m}\mathcal{P}_U \cong j^{*m}\mathcal{S}_U$, then the projective cover in ${}^r\mathbf{Perv}(X) \cong {}^t\mathbf{Perv}(X)$ of the (shifted) simple object ${}^r\mathcal{S}_U \cong j_*\Bbbk_U[2]$ is*

$$ {}^t\mathcal{P}_U \cong {}^p j_{!*} j^{*m}\mathcal{S}_U \cong \Bbbk_X[2]. $$

Moreover, since ${}^m\mathbf{Perv}(X)$ is faithful, the injective hull in ${}^r\mathbf{Perv}(X) \cong {}^t\mathbf{Perv}(X)$ of the (shifted) simple object ${}^r\mathcal{S}_U \cong j_*\Bbbk_U[2]$ is

$$ {}^t\mathcal{I}_U \cong {}^m\mathcal{I}_U \cong j_*\Bbbk_U[2]. $$

Note that, dually to the situation described in Example 5.2.3.2, the object ${}^r\mathcal{S}_U$ is simple and injective.

Corollary 5.2.3.8. *Let $\mathcal{L}$ and $\mathcal{M}$ local systems on strata of U. Then, we have*

$$ \mathrm{Hom}_{{}^q\mathbf{Perv}(X)}({}^q\mathcal{P}_{\mathcal{M}}, {}^q\mathcal{P}_{\mathcal{L}}) \cong \mathrm{Hom}_{{}^p\mathbf{Perv}(U)}(j^{*p}\mathcal{P}_{\mathcal{M}}, j^{*p}\mathcal{P}_{\mathcal{L}}) \cong \mathrm{Hom}_{{}^r\mathbf{Perv}(X)}({}^r\mathcal{P}_{\mathcal{M}}, {}^r\mathcal{P}_{\mathcal{L}}). $$

Proof. It follows immediately from ${}^q\mathcal{P}_{\mathcal{L}} \cong {}^p j_! j^{*p}\mathcal{P}_{\mathcal{L}}$ and ${}^r\mathcal{P}_{\mathcal{L}} \cong {}^p j_{!*} j^{*p}\mathcal{P}_{\mathcal{L}}$ (and similarly for ${}^q\mathcal{P}_{\mathcal{M}}$ and ${}^r\mathcal{P}_{\mathcal{M}}$). □

Chapter 6

Examples and Special Cases

In this final chapter, we study the category of p-perverse sheaves on the projective space $\mathbb{P}^n$ with respect to the affine stratification. In particular, for $n = 2$ we can fully describe the category ${}^p\mathbf{Perv}(\mathbb{P}^2)$ for any GM-perversity p on $\mathbb{P}^2$ and, working inductively, we can extend some results for any n.

In Section 6.1, we recall the case of $\mathbb{P}^1$ for the middle perversity, see Example 2.3.7.4 and [Woo09, Section 3.1]. This case can be regarded as the simplest non-trivial case and the one we aim to generalise.

In Section 6.2, we study perverse sheaves on $X = \mathbb{P}^2$ with the affine stratification. We start by introducing the GM-perversities and noting that, by duality, it is enough to study four perversities which are the middle one, the zero perversity and two more. We then divide the study of the category ${}^p\mathbf{Perv}(\mathbb{P}^2)$ depending on which of the four above perversities we are considering. Each subsection relative to a specific perversity is organised as follows. We start by introducing the simple objects and then we compute the Ext-algebra. This gives a way to identify the Ext-quiver with relations $(\mathrm{Q}_p(X), \mathrm{I}_p(X))$ of the category ${}^p\mathbf{Perv}(\mathbb{P}^2)$. Then, the equivalence of categories ${}^p\mathbf{Perv}(X) \simeq \mathbf{rep}(\mathrm{Q}_p(X), \mathrm{I}_p(X))$ allows us to count and recognise the indecomposable p-perverse sheaves. We list all the indecomposable objects in ${}^p\mathbf{Perv}(X)$ and give a characterisation of them in terms of their quiver representation, a diagrammatic description, as a quiver with a map to $\mathrm{Q}_p(X)$ with one vertex for each element of a basis of the representation and an arrow between vertices when the source basis element maps to the target one under the corresponding image arrow in $\mathrm{Q}_p(X)$ (where there is no arrow the basis element maps to zero), and the dimension vector (which does not uniquely determine a representation if $\mathrm{I}_p(X) \neq 0$, but it is still helpful).

Moreover, we compute minimal projective presentations for simple objects, which can be extended to minimal projective resolutions. Those can be used in order to calculate the Auslander-Reiten translations of simple objects and the global dimension of the category ${}^{p}\mathbf{Perv}(X)$. We then organise this information in the Auslander-Reiten quiver of the category ${}^{p}\mathbf{Perv}(X)$, whose vertices are given by indecomposable perverse sheaves and whose edges are irreducible morphisms. Finally, the remaining Auslander-Reiten translations can be found either using minimal projective presentations and Section 3.5.4 or by general theory, see Theorem 2.2.3.23. Finally we determine if the considered heart is faithful or not using some ad hoc techniques, such as Beilinson's result for algebraic varieties, see [Bei87b], or the existence of some object which is either projective or injective and has higher cohomology on a closed union of strata.

In Section 6.3, we generalise the situation of Section 6.2 by inductively studying p-perverse sheaves on $X = \mathbb{P}^n$ with the affine stratification. We extend some results contained in Section 6.2 to the category ${}^{p}\mathbf{Perv}(\mathbb{P}^n)$, with particular attention to the zero (dually the top) and middle perversities. In the case ${}^{o}\mathbf{Perv}(\mathbb{P}^n) \simeq \mathbf{Constr}(\mathbb{P}^n)$, one recovers the well-known Auslander-Reiten quiver of the category of representations of the $\mathbb{A}_{n+1}$ quiver. For the middle perversity, we give a conjecture for the number of indecomposable objects in ${}^{m}\mathbf{Perv}(\mathbb{P}^n)$ and we explain how we expect the Auslander-Reiten quiver should be.

We end the thesis with Section 6.4, where we pose some questions which might be interesting to explore in future research.

6.1 Complex Projective Line

Let us consider $X = \mathbb{P}^1$ stratified as in Example 2.3.1.6.iii) and let $p = m$ the middle perversity. Let us denote the complementary maps between the two strata by

$$U \cong \mathbb{C} \overset{j}{\hookrightarrow} X \overset{i}{\hookleftarrow} S \cong \{\mathrm{pt}\},$$

then we have $\mathrm{Q}_m(U) \cong \mathrm{Q}_m(S)$ is the quiver with one vertex (we label by 1 the vertex of $\mathrm{Q}_m(U)$ and by 0 the vertex of $\mathrm{Q}_m(S)$), no arrows and no relations. By Remark 4.2.3.13, we have that Ext-quiver of ${}^{m}\mathbf{Perv}(X)$ is

$$\mathrm{Q}_m(\mathbb{P}^1) = \begin{tikzcd} 1 \arrow[r, bend left, "\alpha"] & 0 \arrow[l, bend left, "\beta"] \end{tikzcd}$$

with relation $\mathrm{I}_m(X) = \{\beta \circ \alpha = 0\}$, that is the cycle at the vertex 1 is zero. We want to show that by extending the simple object in $\mathbf{rep}(\mathrm{Q}_m(U))$ and $\mathrm{Q}_m(S)$, we can reach all the indecomposable objects in ${}^m\mathbf{Perv}(X) \simeq \mathbf{rep}(\mathrm{Q}_m(X), \mathrm{I}_m(X))$. The categories ${}^m\mathbf{Perv}(U)$ and ${}^m\mathbf{Perv}(S)$ are semisimple with one non zero simple object. We denote it by S_i for $i = 0, 1$, depending if we consider it in $\mathbf{rep}(\mathrm{Q}_m(S))$ (that is if $i = 0$) or in $\mathbf{rep}(\mathrm{Q}_m(U))$ (if $i = 1$). We then have

$$i_*(S_0) = \begin{tikzcd} 0 \arrow[r, bend left] & \Bbbk \arrow[l, bend left] \end{tikzcd}$$

$${}^p j_!(S_1) = \begin{tikzcd} \Bbbk \arrow[r, bend left, "1"] & \Bbbk \arrow[l, bend left, "0"] \end{tikzcd}$$

$${}^p j_*(S_1) = \begin{tikzcd} \Bbbk \arrow[r, bend left, "0"] & \Bbbk \arrow[l, bend left, "1"] \end{tikzcd}$$

$${}^p j_{!*}(S_1) = \begin{tikzcd} \Bbbk \arrow[r, bend left] & 0 \arrow[l, bend left] \end{tikzcd}$$

$${}^p J(S_1) = \begin{tikzcd} \Bbbk \arrow[r, bend left, "(1\ 0)"] & \Bbbk^2 \arrow[l, bend left, "(0\ 1)^t"] \end{tikzcd}$$

where ${}^pJ$ denotes the maximal extension. Note that in this way one recovers the five indecomposable representations of the indecomposable objects in ${}^m\mathbf{Perv}(\mathbb{P}^1)$, see Figure 6.1.

We now want to study the irreducible maps. One can note that there are maps

$$\begin{aligned} &{}^p j_!(S_1) \twoheadrightarrow {}^p j_{!*}(S_1) \\ &{}^p j_{!*}(S_1) \hookrightarrow {}^p j_*(S_1) \\ &{}^p j_!(S_1) \hookrightarrow {}^p J(S_1) \\ &{}^p J(S_1) \twoheadrightarrow {}^p j_*(S_1), \end{aligned} \tag{6.1}$$

where in particular the first map in (6.1) is the projective cover map and the second the injective hull map. By using the techniques explained in Section 2.2.4, one can check that they are irreducible. Furthermore, one can complete the first two (or dually the second two) maps in (6.1) to get the following short exact sequences in ${}^p\mathbf{Perv}(X)$

$$\begin{aligned} &0 \to i_*\mathcal{S}_0 \to {}^p j_!\mathcal{S}_1 \to {}^p j_{!*}\mathcal{S}_1 \to 0 \\ &0 \to {}^p j_{!*}\mathcal{S}_1 \to {}^p j_*\mathcal{S}_1 \to i_*\mathcal{S}_0 \to 0, \end{aligned}$$

where one can check that all the maps are irreducible. Therefore, one has the following Auslander-Reiten quiver, c.f. Example 2.3.7.4.

$$\begin{tikzcd}
 & & {}^pJ\mathcal{S}_1 \arrow[dr, two heads] & & \\
 & {}^pj_!\mathcal{S}_1 \arrow[ur, hook] \arrow[dr, two heads] & & {}^pj_{!*}\mathcal{S}_1 \arrow[ll, dashed] \arrow[dr, two heads] & \\
i_*\mathcal{S}_0 \arrow[ur, hook] & & {}^pj_{!*}\mathcal{S}_1 \arrow[ll, dashed] \arrow[ur, hook] & & i_*\mathcal{S}_0 \arrow[ll, dashed]
\end{tikzcd}$$

Figure 6.1: Auslander-Reiten quiver of ${}^m\mathbf{Perv}(\mathbb{P}^1)$

6.2 Projective Sphere

In this section, we study p-perverse sheaves on $X = \mathbb{P}^2$ with the affine stratification for any GM-perversity p on X. Let us start by fixing the notation for the strata S_i of X. Let us consider $X = \mathbb{P}^2$ with stratification induced by the affine filtration

$$X_2 = \mathbb{P}^2 \supset X_1 = \mathbb{P}^1 \supset X_0 = \mathbb{P}^0 \supset \emptyset,$$

see Example 2.3.1.6.iii). There are three strata, which we will denote by

$$S_0 \cong \mathbb{P}^0, \qquad S_1 \cong \mathbb{C} \qquad \text{and} \qquad S_2 \cong \mathbb{C}^2$$

respectively. We will use the following maps

$$\begin{tikzcd}
S_2 \arrow[r, hook, "j"] & X & Z = S_1 \cup S_0 \arrow[l, hook', "i"'] & S_0 \arrow[l, hook', "t"'] \\
 & & S_1 \arrow[u, hook, "l"] &
\end{tikzcd} . \tag{6.2}$$

All the above maps are affine inclusions, therefore perverse functors coincide with functors at the level of the constructible derived category when $p \in \{m, o\}$, see Remark 2.3.5.1. Furthermore, $(j, i \circ t)$ and (t, l) are pairs of complementary inclusions.

6.2.1 GM-perversities on $X = \mathbb{P}^2$

We want to study the category ${}^p\mathbf{Perv}(X)$ when $X = \mathbb{P}^2$ and p is a GM-perversity, see Remark 2.3.3.7. Let us denote by $p(S) = (p(S_2), p(S_1), p(S_0))$ a GM-perversity on X, then the possible choices are described in the following diagram.

$$\begin{tikzcd}
& s = (-1,0,0) \arrow[dl, "-(0{,}1{,}1)"'] & & r = (-3,-2,0) \arrow[dr, "+(0{,}1{,}1)"] & \\
o = (0,0,0) & & m = (-2,-1,0) \arrow[ul, "-(0{,}0{,}1)"'] \arrow[dl, "-(0{,}1{,}1)"] \arrow[ur, "+(0{,}0{,}1)"] \arrow[dr, "+(0{,}1{,}1)"'] & & t = (-4,-2,0) \\
& q = (-1,-1,0) \arrow[ul, "-(0{,}0{,}1)"] & & u = (-3,-1,0) \arrow[ur, "+(0{,}0{,}1)"'] &
\end{tikzcd}$$

Figure 6.2: There are seven possible GM-perversities on $X = \mathbb{P}^2$.

Remark 6.2.1.1. *The GM-perversities on $X = \mathbb{P}^2$ of Fugure 6.2 are related by duality. For instance, the middle perversity m is self dual while the pairs (r, s), (q, u) and (o, t) are pairs of dual perversities. Therefore, it is enough to study the category ${}^p\mathbf{Perv}(X)$ for the perversities $p \in \{m, r, s, o\}$.*

Moreover, note that the above perversities can be obtained as nearby perversities as described in Section 5.2. Indeed, one can start from the middle perversity m and add or subtract one on a closed union of strata, which can be either S_0 or $S_1 \cup S_0$. After shifting in order to have $p(S_0) = 0$, the process finishes once one reaches the zero perversity o or dually the top perversity t.

6.2.2 ${}^m\mathbf{Perv}(\mathbb{P}^2)$

Let us consider the middle perversity on $X = \mathbb{P}^2$ given by $m(S_i) = -\dim_{\mathbb{C}} S_i = -i$ for any stratum $S_i \subset X$, that is $m = (-2, -1, 0)$.

Simple Objects

There is one simple object for each stratum $S_i \subset X$. Since the closure of each stratum S_i is a smooth manifold, the three simple objects in ${}^m\mathbf{Perv}(\mathbb{P}^2)$ are (extensions by zero of) shifted constant sheaves of the form $\mathbb{k}_{\mathbb{P}^i}[i]$ for $i = 0, 1, 2$, that is using the maps of (6.2) we have

$$\mathcal{S}_0 = i_* t_* \mathbb{k}_{\mathbb{P}^0}, \qquad \mathcal{S}_1 = i_* \mathbb{k}_{\mathbb{P}^1}[1] \qquad \text{and} \qquad \mathcal{S}_2 = \mathbb{k}_X[2].$$

Ext-Algebra

In order to compute the Ext-algebra, we need to determine the Ext-groups $\mathrm{Ext}^k_{\mathbf{D}_c(X)}(\mathcal{S}_i, \mathcal{S}_j)$ for $i, j = 0, 1, 2$. We will use the following conventions to simplify the notation. We will suppress all the extension by zero from closed union of strata (that is i_* and t_*), but we will always specify the ambient category where we are working in. We will make use of the following identifications:

$$\begin{aligned}
t^*\mathcal{S}_1 &\cong t^*\Bbbk_{\mathbb{P}^1}[1] \cong \Bbbk_{\mathbb{P}^0}[1] \cong \mathcal{S}_0[1],\\
t^!\mathcal{S}_1 &\cong \mathcal{D}t^*\mathcal{S}_1 \cong \mathcal{D}(\Bbbk_{\mathbb{P}^0}[1]) \cong \Bbbk_{\mathbb{P}^0}[-1] \cong \mathcal{S}_0[-1],\\
i^*\mathcal{S}_2 &\cong \Bbbk_{\mathbb{P}^1}[2] \cong \mathcal{S}_1[1],\\
i^!\mathcal{S}_2 &\cong \mathcal{D}i^*\mathcal{S}_2 \cong \mathcal{D}(\Bbbk_{\mathbb{P}^1}[2]) \cong (\mathcal{D}\Bbbk_{\mathbb{P}^1}[1])[1] \cong \Bbbk_{\mathbb{P}^1}[1][-1] \cong \mathcal{S}_1[-1].
\end{aligned}$$

Moreover, for $\mathcal{E} \in \mathbf{D}_c(Y)$ we will use that $\mathrm{Ext}^i_{\mathbf{D}_c(Y)}(\Bbbk_Y, \mathcal{E}) \cong H^i(Y; \mathcal{E})$ to compiute the Ext-groups as cohomology groups.

We then have:

$$\begin{aligned}
&\mathrm{Ext}^i_{\mathbf{D}_c(X)}(\mathcal{S}_0, \mathcal{S}_0) \cong H^i(S_0; \Bbbk) \cong \begin{cases} \Bbbk & \text{if } i = 0 \\ 0 & \text{if } i \neq 0 \end{cases},\\
&\mathrm{Ext}^i_{\mathbf{D}_c(X)}(\mathcal{S}_0, \mathcal{S}_1) \cong \mathrm{Ext}^i_{\mathbf{D}_c(S_0)}(\mathcal{S}_0, t^!\mathcal{S}_1) \cong \mathrm{Ext}^i_{\mathbf{D}_c(S_0)}(\mathcal{S}_0, \mathcal{S}_0[-1]) \cong \begin{cases} \Bbbk & \text{if } i = 1 \\ 0 & \text{if } i \neq 1 \end{cases},\\
&\mathrm{Ext}^i_{\mathbf{D}_c(X)}(\mathcal{S}_0, \mathcal{S}_2) \cong \mathrm{Ext}^i_{\mathbf{D}_c(Z)}(\mathcal{S}_0, i^!\mathcal{S}_2) \cong \mathrm{Ext}^i_{\mathbf{D}_c(Z)}(\mathcal{S}_0, \mathcal{S}_1[-1]) \cong \mathrm{Ext}^i_{\mathbf{D}_c(S_0)}(\mathcal{S}_0, t^!\mathcal{S}_1[-1])\\
&\qquad\qquad \cong \mathrm{Ext}^i_{\mathbf{D}_c(S_0)}(\mathcal{S}_0, \mathcal{S}_0[-2]) \cong \begin{cases} \Bbbk & \text{if } i = 2 \\ 0 & \text{if } i \neq 2 \end{cases},\\
&\mathrm{Ext}^i_{\mathbf{D}_c(X)}(\mathcal{S}_1, \mathcal{S}_0) \cong \mathrm{Ext}^i_{\mathbf{D}_c(S_0)}(t^*\mathcal{S}_1, \mathcal{S}_0) \cong \mathrm{Ext}^i_{\mathbf{D}_c(S_0)}(\mathcal{S}_0[1], \mathcal{S}_0) \cong \begin{cases} \Bbbk & \text{if } i = 1 \\ 0 & \text{if } i \neq 1 \end{cases},\\
&\mathrm{Ext}^i_{\mathbf{D}_c(X)}(\mathcal{S}_1, \mathcal{S}_1) \cong H^i(Z; \Bbbk) \cong \begin{cases} \Bbbk & \text{if } i = 0, 2 \\ 0 & \text{otherwise} \end{cases},\\
&\mathrm{Ext}^i_{\mathbf{D}_c(X)}(\mathcal{S}_1, \mathcal{S}_2) \cong \mathrm{Ext}^i_{\mathbf{D}_c(Z)}(\mathcal{S}_1, i^!\mathcal{S}_2) \cong \mathrm{Ext}^i_{\mathbf{D}_c(Z)}(\mathcal{S}_1, \mathcal{S}_1[-1]) \cong \begin{cases} \Bbbk & \text{if } i = 1 \\ 0 & \text{if } i \neq 1 \end{cases},
\end{aligned}$$

$$\operatorname{Ext}^i_{\mathbf{D}_c(X)}(\mathcal{S}_2,\mathcal{S}_0) \cong \operatorname{Ext}^i_{\mathbf{D}_c(Z)}(i^*\mathcal{S}_2,\mathcal{S}_0) \cong \operatorname{Ext}^i_{\mathbf{D}_c(Z)}(\mathcal{S}_1[1],\mathcal{S}_0) \cong \operatorname{Ext}^i_{\mathbf{D}_c(S_0)}(t^*\mathcal{S}_1[1],\mathcal{S}_0)$$
$$\cong \operatorname{Ext}^i_{\mathbf{D}_c(S_0)}(\mathcal{S}_0[2],\mathcal{S}_0) \cong \begin{cases} \Bbbk & \text{if } i=2 \\ 0 & \text{if } i \neq 2 \end{cases},$$

$$\operatorname{Ext}^i_{\mathbf{D}_c(X)}(\mathcal{S}_2,\mathcal{S}_1) \cong \operatorname{Ext}^i_{\mathbf{D}_c(Z)}(i^*\mathcal{S}_2,\mathcal{S}_1) \cong \operatorname{Ext}^i_{\mathbf{D}_c(Z)}(\mathcal{S}_1[1],\mathcal{S}_1) \cong \begin{cases} \Bbbk & \text{if } i=1 \\ 0 & \text{if } i \neq 1 \end{cases},$$

$$\operatorname{Ext}^i_{\mathbf{D}_c(X)}(\mathcal{S}_2,\mathcal{S}_2) \cong H^i(X;\Bbbk) \cong \begin{cases} \Bbbk & \text{if } i=0,2,4 \\ 0 & \text{otherwise} \end{cases}.$$

Ext-Quiver and Relations

The Ext-quiver $\mathrm{Q}_m(X)$ of the category ${}^m\mathbf{Perv}(X)$ can be constructed by considering the Ext^1-groups between simple objects. Therefore, we have

$$\begin{tikzcd} 2 \arrow[r, bend left, "\alpha"] & 1 \arrow[l, bend left, "\delta"] \arrow[r, bend left, "\beta"] & 0 \arrow[l, bend left, "\gamma"] \end{tikzcd}.$$

On the other hand, the relations $\mathrm{I}_m(\mathbb{P}^2)$ arise from Ext^2-groups between simple objects. We have that $\mathrm{I}_m(\mathbb{P}^2)$ is generated by (at most) four relations in $e_i\mathrm{I}_m(\mathbb{P}^2)e_j$, where $i=j=1$, or $i=j=2$, or $i=0$ and $j=2$ or $i=2$ and $j=0$. By Lemma 4.2.3.8 we know that $\delta\circ\alpha \in e_2\mathrm{I}_m(\mathbb{P}^2)e_2$. For the same reason, $e_1\mathrm{I}_m(\mathbb{P}^2)e_1 = \langle\gamma\circ\beta\rangle$. By (4.5) and Lemma 4.2.3.2 $\gamma\circ\beta \in e_1\mathrm{I}_m(\mathbb{P}^2)e_1$. By (4.5) and Lemma 4.2.3.2 we know that

$$\beta\circ\alpha + \{\text{paths of length } {>}2\} \in e_2\mathrm{I}_m(\mathbb{P}^2)e_0.$$

However, all paths from 2 to 0 of length greater than two are already in $\mathrm{I}_m(\mathbb{P}^2)$, therefore $\beta\circ\alpha \in \mathrm{I}_m(\mathbb{P}^2)$. A similar argument shows that $\gamma\circ\delta \in \mathrm{I}_m(\mathbb{P}^2)$. Hence, the relations are

$$\delta\circ\alpha = 0, \quad \delta\circ\gamma = 0, \quad \beta\circ\alpha = 0 \quad \text{and} \quad \gamma\circ\beta = 0.$$

That is, the relations are given by the clockwise length two cycles around the vertices 2 and 1 and the two length two paths from 2 to 0 and from 0 to 2. Note that the relations are quadratic.

Indecomposable Objects

The equivalence of categories ${}^{m}\mathbf{Perv}(\mathbb{P}^2) \simeq \mathbf{rep}(\mathrm{Q}_m(\mathbb{P}^2), \mathrm{I}_m(\mathbb{P}^2))$ gives an explicit way to list all the indecomposable perverse sheaves, which we can equivalently characterise as irreducible representations of the Ext-quiver with relations $(\mathrm{Q}_m(X), \mathrm{I}_m(X))$ and diagrammatically, as a quiver with a map to $\mathrm{Q}_m(\mathbb{P}^2)$ with one vertex for each element of a basis of the representation and an arrow between vertices when the source basis element maps to the target one under the corresponding image arrow in $\mathrm{Q}_m(\mathbb{P}^2)$. We also give the corresponding dimension vector of each irreducible quiver representation.

Object	Quiver Representation	Path	Dimension Vector
$\mathcal{S}_0$	$0 \rightleftarrows 0 \rightleftarrows \Bbbk$	$\times \quad \times \quad \bullet$	(001)
$\mathcal{S}_1$	$0 \rightleftarrows \Bbbk \rightleftarrows 0$	$\times \quad \bullet \quad \times$	(010)
$\mathcal{S}_2$	$\Bbbk \rightleftarrows 0 \rightleftarrows 0$	$\bullet \quad \times \quad \times$	(100)
$\mathcal{P}_2$	$\Bbbk \underset{0}{\overset{1}{\rightleftarrows}} \Bbbk \rightleftarrows 0$	$\bullet \rightarrow \bullet \quad \times$	(110)
$\mathcal{I}_2$	$\Bbbk \underset{1}{\overset{0}{\rightleftarrows}} \Bbbk \rightleftarrows 0$	$\bullet \leftarrow \bullet \quad \times$	(110)
$\widehat{\mathcal{P}}_1$	$0 \rightleftarrows \Bbbk \underset{0}{\overset{1}{\rightleftarrows}} \Bbbk$	$\times \quad \bullet \rightarrow \bullet$	(011)
$\widehat{\mathcal{I}}_1$	$0 \rightleftarrows \Bbbk \underset{1}{\overset{0}{\rightleftarrows}} \Bbbk$	$\times \quad \bullet \leftarrow \bullet$	(011)
$\overline{\mathcal{P}}_2$	$\Bbbk \underset{(01)^t}{\overset{(10)}{\rightleftarrows}} \Bbbk^2 \rightleftarrows 0$	$\bullet \rightleftarrows \substack{\bullet \\ \bullet} \quad \times$	(120)
$\mathcal{P}_0 \cong \mathcal{I}_0$	$0 \rightleftarrows \Bbbk \underset{(01)^t}{\overset{(10)}{\rightleftarrows}} \Bbbk^2$	$\times \quad \bullet \rightleftarrows \substack{\bullet \\ \bullet}$	(012)
$\mathcal{M}_2$	$\Bbbk \underset{1}{\overset{0}{\rightleftarrows}} \Bbbk \underset{0}{\overset{1}{\rightleftarrows}} \Bbbk$	$\bullet \leftarrow \bullet \rightarrow \bullet$	(111)
$\mathcal{N}_2$	$\Bbbk \underset{0}{\overset{1}{\rightleftarrows}} \Bbbk \underset{1}{\overset{0}{\rightleftarrows}} \Bbbk$	$\bullet \rightarrow \bullet \leftarrow \bullet$	(111)
$\mathcal{P}_1$	$\Bbbk \underset{(01)^t}{\overset{(10)}{\rightleftarrows}} \Bbbk^2 \underset{(00)}{\overset{(01)^t}{\rightleftarrows}} \Bbbk$	$\bullet \rightleftarrows \substack{\bullet \\ \bullet} \rightarrow \bullet$	(121)
$\mathcal{I}_1$	$\Bbbk \underset{(01)^t}{\overset{(10)}{\rightleftarrows}} \Bbbk^2 \underset{(10)}{\overset{(00)^t}{\rightleftarrows}} \Bbbk$	$\bullet \rightleftarrows \substack{\bullet \\ \bullet} \leftarrow \bullet$	(121)
$\mathcal{L}_2$	$\Bbbk \underset{(01)^t}{\overset{(10)}{\rightleftarrows}} \Bbbk^2 \underset{\left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right)}{\overset{\left(\begin{smallmatrix}0&0\\0&1\end{smallmatrix}\right)}{\rightleftarrows}} \Bbbk^2$	$\bullet \rightleftarrows \substack{\bullet \\ \bullet} \rightleftarrows \substack{\bullet \\ \bullet}$	(122)

Table 6.1: Indecomposable perverse sheaves in ${}^m\mathbf{Perv}(\mathbb{P}^2)$.

Minimal Projective Presentations and Global Dimension

We now consider minimal projective presentations of simple objects in ${}^m\mathbf{Perv}(\mathbb{P}^2)$. Recall that if $\mathcal{S}_i \in {}^p\mathbf{Perv}(X)$ is a simple perverse sheaf, a minimal projective presentation is given by

$$\mathcal{P}(\ker \pi_i) \to \mathcal{P}_i \overset{\pi_i}{\twoheadrightarrow} \mathcal{S}_i,$$

where $\mathcal{P}_i \in {}^p\mathbf{Perv}(X)$ is the projective cover of $\mathcal{S}_i$ with projective cover map π_i and $\mathcal{P}(\mathcal{E})$ denotes the projective cover of the object $\mathcal{E} \in {}^p\mathbf{Perv}(X)$. Since we have

$$\ker(\pi_i) \cong \mathrm{rad}(\mathcal{P}_i) \cong \begin{cases} \widehat{\mathcal{P}}_1 & \text{if } i = 0 \\ \mathcal{S}_0 \oplus \mathcal{P}_2 & \text{if } i = 1 \\ \mathcal{S}_1 & \text{if } i = 2 \end{cases}, \tag{6.3}$$

and $\mathcal{P}(\widehat{\mathcal{P}}_1) \cong \mathcal{P}_1$, the minimal projective presentations of simple objects are

$$\begin{aligned} &\mathcal{P}_1 \to \mathcal{P}_0 \twoheadrightarrow \mathcal{S}_0 \\ &\mathcal{P}_2 \oplus \mathcal{P}_0 \to \mathcal{P}_1 \twoheadrightarrow \mathcal{S}_1 \\ &\mathcal{P}_1 \to \mathcal{P}_2 \twoheadrightarrow \mathcal{S}_2. \end{aligned}$$

The minimal projective presentation can be extended inductively to obtain minimal projective resolutions, see Section 3.5.1. Therefore, we have

$$\begin{aligned} &\mathcal{P}_0^\bullet = \mathcal{P}_2 \hookrightarrow \mathcal{P}_1 \to \mathcal{P}_0 \\ &\mathcal{P}_1^\bullet = \mathcal{P}_2 \hookrightarrow \mathcal{P}_1 \to \mathcal{P}_0 \oplus \mathcal{P}_2 \to \mathcal{P}_1 \\ &\mathcal{P}_2^\bullet = \mathcal{P}_2 \hookrightarrow \mathcal{P}_1 \to \mathcal{P}_0 \oplus \mathcal{P}_2 \to \mathcal{P}_1 \to \mathcal{P}_2. \end{aligned} \tag{6.4}$$

We can conclude that $\mathrm{gldim}({}^m\mathbf{Perv}(X)) = 4$.

AR-quiver

We now want to determine the Auslander-Reiten quiver of the category ${}^m\mathbf{Perv}(\mathbb{P}^2)$. The vertices are given by indecomposable perverse sheaves, therefore they are objects appearing in the list of Table 6.1. We will proceed inductively by adding the open stratum S_2 to $\mathbb{P}^1$. That is, we can assume that we know the Auslander-Reiten quiver of ${}^m\mathbf{Perv}(\mathbb{P}^1)$. The new objects in ${}^m\mathbf{Perv}(\mathbb{P}^2)$ supported on S_2 are the extensions of $\mathcal{S}_2 \in {}^m\mathbf{Perv}(S_2)$ first to $S_2 \cup S_1$ and then the extensions of the obtained objects to $X = S_2 \cup S_1 \cup S_0$. The first step gives rise to the four extensions of $\mathcal{S}_2$ of Figure 6.1). The extension of the four objects supported on S_2 in ${}^m\mathbf{Perv}(\mathbb{P}^1)$ to ${}^m\mathbf{Perv}(X)$ gives rise to the following nine objects of ${}^m\mathbf{Perv}(X)$ supported on S_2.

$$\begin{tikzcd}[column sep=small]
 & & \substack{\overline{\mathcal{P}}_2 \\ (120)} \arrow[dr, hook] & & \\
 & \substack{\mathcal{P}_1 \\ (121)} \arrow[ur, two heads] \arrow[dr, hook] & & \substack{\mathcal{I}_1 \\ (121)} \arrow[dr, two heads] & \\
\substack{\mathcal{P}_2 \\ (110)} \arrow[ur, hook] \arrow[dr, hook] & & \substack{\mathcal{L}_2 \\ (122)} \arrow[ur, two heads] \arrow[dr, two heads] & & \substack{\mathcal{I}_2 \\ (110)} \\
 & \substack{\mathcal{N}_2 \\ (111)} \arrow[ur, hook] \arrow[dr, two heads] & & \substack{\mathcal{M}_2 \\ (111)} \arrow[ur, two heads] & \\
 & & \substack{\mathcal{S}_2 \\ (100)} \arrow[ur, hook] & &
\end{tikzcd}$$

Figure 6.3: Perverse sheaves in ${}^m\mathbf{Perv}(\mathbb{P}^2)$ supported on S_2.

We claim that all the morphisms in Figure 6.3 are irreducible, and are all the irreducible morphisms.

Since $\mathrm{rad}(\mathcal{P}_1) \cong \mathcal{S}_0 \oplus \mathcal{P}_2$, the morphism $\mathcal{P}_2 \hookrightarrow \mathcal{P}_1$ is irreducible by Lemma 2.2.3.6. Moreover, let us consider the short exact sequence

$$0 \to \mathcal{P}_2 \to \mathcal{P}_1 \to \overline{\mathcal{P}}_2 \to 0$$

in ${}^m\mathbf{Perv}(\mathbb{P}^2)$. In order to check if the map $\alpha : \mathcal{P}_1 \twoheadrightarrow \overline{\mathcal{P}}_2$ is irreducible, we apply Lemma 2.2.3.7. That is, we need to check that for any morphism $\tau : \mathcal{P}_2 \to \mathcal{E}$ either there exists $\tau_1 : \mathcal{P}_1 \to \mathcal{E}$ such that $\tau = \tau_1 \circ \alpha$ or there exists $\tau_2 : \mathcal{E} \to \mathcal{P}_1$ such that $\alpha = \tau_2 \circ \tau$

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{P}_2 \arrow[r, "\alpha"] \arrow[d, "\tau"'] & \mathcal{P}_1 \arrow[r] \arrow[dl, shift left, "\tau_1"] & \overline{\mathcal{P}}_2 \arrow[r] & 0 \\
 & \mathcal{E} \arrow[ur, shift left, "\tau_2"] & & &
\end{tikzcd} . \tag{6.5}$$

Since $\mathcal{P}_2 \in {}^m\mathbf{Perv}(X)$ is initial among the objects in ${}^m\mathbf{Perv}(X)$ supported on S_2, the object $\mathcal{E}$ can be any of the ones appearing in Figure 6.3. The fact that $\alpha : \mathcal{P}_2 \hookrightarrow \mathcal{P}_1$ is irreducible is enough to guarantee the existence of either τ_1 or τ_2 with the required property. Therefore, the morphism $\beta : \mathcal{P}_1 \twoheadrightarrow \overline{\mathcal{P}}_2$ is irreducible.

Let us consider the short exact sequence $0 \to \mathcal{P}_2 \xrightarrow{\beta} \mathcal{N}_2 \xrightarrow{\gamma} \mathcal{S}_2 \to 0$ in ${}^m\mathbf{Perv}(\mathbb{P}^2)$. In order to show that the morphism $\beta : \mathcal{P}_2 \hookrightarrow \mathcal{N}_2$ is irreducible, we apply Lemma 2.2.3.7. We need to show that for any morphism $\tau : \mathcal{E} \to \mathcal{S}_2$ either there exists $\tau_1 : \mathcal{E} \to \mathcal{N}_2$ such that $\tau = \gamma \circ \tau_1$ or there exists $\tau_2 : \mathcal{N}_2 \to \mathcal{E}$ such that $\gamma = \tau \circ \tau_1$:

$$\begin{tikzcd}
 & & & \mathcal{E} \arrow[dl, shift right, "\tau_2"'] \arrow[d, "\tau"] & \\
0 \arrow[r] & \mathcal{P}_2 \arrow[r, "\beta"'] & \mathcal{N}_2 \arrow[ur, shift right, "\tau_1"'] \arrow[r, "\gamma'"'] & \mathcal{S}_2 \arrow[r] & 0
\end{tikzcd} \qquad (6.6)$$

In this case, τ cannot be a monomorphism as $\mathcal{S}_2$ is a simple object. The only objects which admit a map into $\mathcal{S}_2$ in ${}^m\mathbf{Perv}(X)$ are $\mathcal{E} \in \{\mathcal{P}_2, \mathcal{N}_2, \mathcal{S}_2\}$, that is $\mathcal{E} \cong P_!(\mathcal{F})$ for some $\mathcal{F} \in {}^m\mathbf{Perv}(X)$ supported on S_2. Then, we have

$$\mathcal{E} \cong \begin{cases} \mathcal{P}_2 & \Rightarrow \ \tau = \gamma \circ \beta \ \text{ and } \ \tau_1 = \beta \\ \mathcal{N}_2 & \Rightarrow \ \tau = \gamma \text{ and } \ \tau_1 = \mathrm{id}_{\mathcal{N}_2} \\ \mathcal{S}_2 & \Rightarrow \ \tau = \mathrm{id}_{\mathcal{S}_2} \ \text{ and } \ \tau_2 = \gamma \end{cases}.$$

Therefore the morphism $\beta : \mathcal{P}_2 \hookrightarrow \mathcal{N}_2$ is irreducible. In order to check that γ is irreducible, we need to show something similar to (6.5). That is, we need to check that for any morphism $\tau : \mathcal{P}_2 \to \mathcal{E}$ in ${}^m\mathbf{Perv}(X)$ either $\exists \tau_1 : \mathcal{N}_2 \to \mathcal{E}$ such that $\tau = \tau_1 \circ \beta$ or that $\exists \tau_2 : \mathcal{E} \to \mathcal{N}_2$ such that $\beta = \tau_2 \circ \tau$:

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{P}_2 \arrow[r, "\beta"] \arrow[d, "\tau"'] & \mathcal{N}_2 \arrow[r, "\gamma"] \arrow[dl, shift left, "\tau_1"] & \mathcal{S}_2 \arrow[r] & 0 \\
 & \mathcal{E} \arrow[ur, shift left, "\tau_2"] & & &
\end{tikzcd}.$$

Again, we have that $\mathcal{E}$ can be any of the objects appearing in Figure 6.3, but the fact that β is irreducible guarantees that there exists either τ_1 or τ_2 with the required property. Therefore, the morphism $\gamma : \mathcal{N}_2 \twoheadrightarrow \mathcal{S}_2$ is irreducible.

Let us consider the short exact sequence in ${}^m\mathbf{Perv}(X)$ induced by the morphism $\delta : \mathcal{P}_1 \hookrightarrow \mathcal{L}_2$, that is $0 \to \mathcal{P}_1 \xrightarrow{\delta} \mathcal{L}_2 \xrightarrow{\sigma} \mathcal{S}_0 \to 0$. In order to check if the morphism δ is irreducible, we need to show something similar to (6.6). That is, we need to prove that any morphism $\tau : \mathcal{E} \to \mathcal{S}_0$ either there exists $\tau_1 : \mathcal{E} \to \mathcal{L}_2$ such that $\tau = \tau \circ \tau_1$ or there exists $\tau_2 : \mathcal{L}_2 \to \mathcal{E}$ such that $\sigma = \tau \circ \tau_1$:

$$\begin{tikzcd}
& & & \mathcal{E} \arrow[d, "\tau"] \arrow[dl, shift right, "\tau_2"'] & \\
0 \arrow[r] & \mathcal{P}_1 \arrow[r, "\delta"'] & \mathcal{L}_2 \arrow[r, "\sigma"'] \arrow[ur, shift right, "\tau_1"'] & \mathcal{S}_0 \arrow[r] & 0
\end{tikzcd}.$$

In order to have a morphism $\tau : \mathcal{E} \to \mathcal{S}_0$, the object $\mathcal{E} \in {}^m\mathbf{Perv}(X)$ has either a sub-object or a quotient supported on S_0. Since $\mathcal{L}_2 \in {}^m\mathbf{Perv}(X)$ is the 'biggest' object and any other object has either a map in or a map out of it, the quotient (or sub-object) of $\mathcal{E}$ supported on S_0 appears as a quotient (or as a sub-object) of $\mathcal{L}_2$. Then, the existence either of τ_1 or τ_2 with the required property is guaranteed.

Let us consider the short exact sequence in ${}^m\mathbf{Perv}(X)$ induced by the morphism $\epsilon : \mathcal{N}_2 \hookrightarrow \mathcal{L}_2$, that is $0 \to \mathcal{N}_2 \xrightarrow{\epsilon} \mathcal{L}_2 \xrightarrow{\theta} \mathcal{S}_0 \to 0$. The above argument applied to the pair of morphisms (ϵ, δ) shows that ϵ is irreducible.

Finally, by duality we have that all the other morphisms in Figure 6.3 are irreducible as they are dual to the ones considered.

We now need to connect the square of Figure 6.3 to ${}^m\mathbf{Perv}(X \smallsetminus S_2) \simeq {}^m\mathbf{Perv}(\mathbb{P}^1)$. Since $\operatorname{rad}(\mathcal{P}_1) \cong \mathcal{P}_2 \oplus \mathcal{S}_0$, we have that the morphism $\mathcal{S}_0 \hookrightarrow \mathcal{P}_1$ is irreducible. For the same reason, since $\operatorname{rad}(\mathcal{P}_2) \cong \mathcal{S}_1$, the morphism $\mathcal{S}_1 \hookrightarrow \mathcal{P}_2$ is irreducible. Finally, we claim that the morphism $\theta : \widehat{\mathcal{I}}_1 \hookrightarrow \mathcal{N}_2$ is also irreducible. Let us consider the short exact sequence induced by θ in ${}^m\mathbf{Perv}(\mathbb{P}^2)$, that is

$$0 \to \widehat{\mathcal{I}}_1 \to \mathcal{N}_2 \xrightarrow{\theta'} \mathcal{S}_2 \to 0.$$

By Lemma 2.2.3.7, we have that θ is irreducible if for any $\tau : \mathcal{E} \to \mathcal{S}_2$ either there exists $\tau_1 : \mathcal{E} \to \mathcal{N}_2$ such that $\tau = \theta' \circ \tau_1$ or there exists $\tau_2 : \mathcal{N}_2 \to \mathcal{E}$ such that $\theta' = \tau \circ \tau_1$, that is

$$\begin{tikzcd}
& & & \mathcal{E} \arrow[d, "\tau"] \arrow[dl, shift right, "\tau_2"'] & \\
0 \arrow[r] & \widehat{\mathcal{I}}_1 \arrow[r, "\theta"] & \mathcal{N}_2 \arrow[r, "\theta'"] \arrow[ur, shift right, "\tau_1"'] & \mathcal{S}_2 \arrow[r] & 0
\end{tikzcd}$$

The same argument applied for (6.6) shows that θ is irreducible. By duality, we have that the morphisms $\mathcal{I}_1 \twoheadrightarrow \mathcal{S}_0$, $\mathcal{I}_2 \twoheadrightarrow \mathcal{S}_1$ and $\mathcal{M}_2 \twoheadrightarrow \widehat{\mathcal{P}}_1$ are irreducible as well. We can draw the Auslander-Reiten quiver of ${}^m\mathbf{Perv}(\mathbb{P}^2)$.

$$
\begin{tikzcd}
 & = & \overset{\mathcal{S}_0}{(001)} \arrow[dr, hook] & & \overset{\overline{\mathcal{P}}_2}{(120)} \arrow[ll, dashed] \arrow[dr] & & \overset{\mathcal{S}_0}{(001)} \arrow[ll, dashed] & = & \\
 & & & \overset{\mathcal{P}_1}{(121)} \arrow[ur, two heads] \arrow[dr, hook] & & \overset{\mathcal{I}_1}{(121)} \arrow[ll, dashed] \arrow[ur, two heads] \arrow[dr, two heads] & & & \\
 & & \overset{\mathcal{P}_2}{(110)} \arrow[ur, hook] \arrow[dr, hook] & & \overset{\mathcal{L}_2}{(122)} \arrow[ll, dashed] \arrow[ur, two heads] \arrow[dr, two heads] & & \overset{\mathcal{I}_2}{(110)} \arrow[ll, dashed] \arrow[dr, two heads] & & \\
= & \overset{\mathcal{S}_1}{(010)} \arrow[ur, hook] \arrow[dr, hook] & & \overset{\mathcal{N}_2}{(111)} \arrow[ll, dashed] \arrow[ur, hook] \arrow[dr, two heads] & & \overset{\mathcal{M}_2}{(111)} \arrow[ll, dashed] \arrow[ur, two heads] \arrow[dr, two heads] & & \overset{\mathcal{S}_1}{(010)} \arrow[ll, dashed] & = \\
 & {} & \overset{\widehat{\mathcal{I}}_1}{(011)} \arrow[l, dashed] \arrow[ur, hook] & & \overset{\mathcal{S}_2}{(100)} \arrow[ll, dashed] \arrow[ur, hook] & & \overset{\widehat{\mathcal{P}}_1}{(011)} \arrow[ll, dashed] \arrow[ur] \arrow[dr, hook] & {} \arrow[l, dashed] & \\
= & \overset{\mathcal{P}_0}{(012)} \arrow[ur, two heads] & & & & & & \overset{\mathcal{I}_0}{(012)} & =
\end{tikzcd}
$$

Figure 6.4: The Auslander-Reiten quiver of the category ${}^m\mathbf{Perv}(\mathbb{P}^2)$.

Note that the square of extensions of objects supported on S_2 of Figure 6.3 appears in the middle of Figure 6.4, while the objects which are extension by zero of ${}^m\mathbf{Perv}(S_0 \cup S_1) \simeq {}^m\mathbf{Perv}(\mathbb{P}^1)$ in ${}^m\mathbf{Perv}(\mathbb{P}^2)$ appear twice, once on the left and once of the right of the middle square, in order to consider irreducible morphisms in and out.

We now show that the ones in Figure 6.4 are all the irreducible morphisms. One can find the Auslander-Reiten translation (and its inverse) of $\mathcal{S}_0 \in {}^m\mathbf{Perv}(\mathbb{P}^2)$ by using Section 3.5.4. Consider the minimal projective resolution of $\mathcal{S}_0$ in (6.11), then we have that the Auslander-Reiten translation of $\mathcal{S}_0$ sits in the exact sequence

$$0 \to \tau(\mathcal{S}_0) \to \mathcal{D}(\mathcal{P}_1)^t \to \mathcal{D}(\mathcal{P}_0)^t \to \mathcal{D}(\mathcal{S}_0)^t \to 0$$

which becomes

$$0 \to \tau(\mathcal{S}_0) \to \mathcal{I}_1 \to \mathcal{I}_2 \to \widehat{\mathcal{I}}_1 \to 0,$$

therefore $\tau(\mathcal{S}_0) \cong \overline{\mathcal{P}}_2$. Similarly, by considering a minimal injective resolution of $\mathcal{S}_0$, one has that $\tau^{-1}(\mathcal{S}_0) \cong \overline{\mathcal{P}}_2$. Using the fact that the radical of $\mathcal{P}_1$ splits as two direct summands, then there are two irreducible maps into $\mathcal{P}_1$ and two irreducible maps out of

$\mathcal{P}_1$. The same argument can by applied to $\mathcal{I}_1$ and its socle to conclude that $\tau(\mathcal{I}_1) \cong \mathcal{P}_1$ by Theorem 2.2.3.23 and Lemma 2.2.3.24. If one keeps applying this argument, one finds all the Auslander-Reiten translations and can then conclude that the maps in Figure 6.4 are all the irreducible morphisms between indecomposable objects in ${}^m\mathbf{Perv}(\mathbb{P}^2)$.

Remark 6.2.2.1. *One could instead add the closed stratum S_0 to $U \cong S_2 \cup S_1$. In that case, in order to draw the Auslander-Reiten quiver, one needs to find the extensions of $\mathcal{S}_0 \in {}^m\mathbf{Perv}(S_0)$ to X. They are*

$$\mathcal{S}_0 \hookrightarrow \mathcal{P}_1 \hookrightarrow \mathcal{L}_2 \twoheadrightarrow \mathcal{M}_2 \twoheadrightarrow \widehat{\mathcal{P}}_1 \hookrightarrow \mathcal{I}_0$$

and dually

$$\mathcal{P}_0 \twoheadrightarrow \widehat{\mathcal{I}}_1 \hookrightarrow \mathcal{N}_2 \hookrightarrow \mathcal{L}_2 \twoheadrightarrow \mathcal{I}_1 \twoheadrightarrow \mathcal{S}_0.$$

Then, one needs to check that the above chain of morphisms are compositions of irreducible morphisms. Finally, one needs to show that the extensions of $\mathcal{S}_0$ to X are connected to ${}^m\mathbf{Perv}(U)$ by the irreducible morphisms $\mathcal{S}_1 \hookrightarrow \widehat{\mathcal{I}}_1$, $\mathcal{P}_2 \hookrightarrow \mathcal{N}_2$, $\overline{\mathcal{P}}_2 \hookrightarrow \mathcal{I}_1$ and $\mathcal{I}_1 \twoheadrightarrow \mathcal{I}_2$ (and their duals). Of course the result will be the same as the one showed in Figure 6.4.

On Faithfulness

The heart ${}^m\mathbf{Perv}(\mathbb{P}^2) \subset \mathbf{D}_c(\mathbb{P}^2)$ is faithful by [Bei87b].

6.2.3 ${}^q\mathbf{Perv}(\mathbb{P}^2)$

Let us consider the perversity $q = (-1, -1, 0)$ obtained by subtracting $(0, 1, 1)$ from the middle perversity (and then shifting in order to have $q(S_2) = 0$).

Simple Objects

The three simple objects in ${}^q\mathbf{Perv}(\mathbb{P}^2)$ can be determined by using Proposition 5.2.2.3. They are

$$\mathcal{S}_0 = i_* t_* \Bbbk_{\mathbb{P}^0}, \qquad \mathcal{S}_1 = i_* \Bbbk_{\mathbb{P}^1}[1] \qquad \text{and} \qquad \mathcal{S}_2 = j_! \Bbbk_{\mathbb{C}^2}[1].$$

Alternatively, one can check that they satisfy the strong conditions of Remark 2.3.5.5.iii). In particular it is clear that the former object is q-perverse, while for the latter one can note that

$$i^* j_! \Bbbk_{\mathbb{C}^2}[1] = 0 \quad \text{and} \quad i^! j_! \Bbbk_{\mathbb{C}^2}[1] \cong i^* j_* \Bbbk_{\mathbb{C}^2} \cong \Bbbk_{\mathbb{P}^1} \oplus \Bbbk_{\mathbb{P}^1}[-1].$$

Ext-Algebra

We compute the Ext-groups $\mathrm{Ext}^k_{\mathbf{D}_c(X)}(\mathcal{S}_i,\mathcal{S}_j)$ for $i,j=0,1,2$ in order to determine the Ext-algebra. Note that the simple objects $\mathcal{S}_0,\mathcal{S}_1\in{}^q\mathbf{Perv}(\mathbb{P}^2)$ agree with the corresponding ones in ${}^m\mathbf{Perv}(\mathbb{P}^2)$. Therefore, we only need to calculate the Ext^k-groups when the simple object $\mathcal{S}_2$ is involved. We have:

$$\begin{aligned}\mathrm{Ext}^i_{\mathbf{D}_c(X)}(\mathcal{S}_0,\mathcal{S}_2) &\cong \mathrm{Ext}^i_{\mathbf{D}_c(X)}(i_*t_*\Bbbk_{\mathbb{P}^0}, j_!\Bbbk_{\mathbb{C}^2}[1]) \cong \mathrm{Ext}^i_{\mathbf{D}_c(Z)}(t_*\Bbbk_{\mathbb{P}^0}, i^!j_!\Bbbk_{\mathbb{C}^2}[1])\\ &\cong \mathrm{Ext}^i_{\mathbf{D}_c(S_0)}(\Bbbk_{\mathbb{P}^0}, t^!(\Bbbk_{\mathbb{P}^1}\oplus\Bbbk_{\mathbb{P}^1}[-1]))\\ &\cong \mathrm{Ext}^i_{\mathbf{D}_c(S_0)}(\Bbbk_{\mathbb{P}^0}, \Bbbk_{\mathbb{P}^0}[-2]\oplus\Bbbk_{\mathbb{P}^0}[-3]) \cong \begin{cases}\Bbbk & \text{if } i=2,3\\ 0 & \text{otherwise}\end{cases},\end{aligned}$$

$$\begin{aligned}\mathrm{Ext}^i_{\mathbf{D}_c(X)}(\mathcal{S}_1,\mathcal{S}_2) &\cong \mathrm{Ext}^i_{\mathbf{D}_c(X)}(i_*\Bbbk_{\mathbb{P}^1}[1], j_!\Bbbk_{\mathbb{C}^2}[1]) \cong \mathrm{Ext}^i_{\mathbf{D}_c(Z)}(\Bbbk_{\mathbb{P}^1}[1], i^!j_!\Bbbk_{\mathbb{C}^2}[1])\\ &\cong \mathrm{Ext}^i_{\mathbf{D}_c(Z)}(\Bbbk_{\mathbb{P}^1}[1], \Bbbk_{\mathbb{P}^1}\oplus\Bbbk_{\mathbb{P}^1}[-1]) \cong \begin{cases}\Bbbk & \text{if } i=1,2,3,4\\ 0 & \text{otherwise}\end{cases},\\ \mathrm{Ext}^i_{\mathbf{D}_c(X)}(\mathcal{S}_2,\mathcal{S}_0) &\cong \mathrm{Ext}^i_{\mathbf{D}_c(X)}(j_!\Bbbk_{\mathbb{C}^2}[1], i_*t_*\Bbbk_{\mathbb{P}^0}) \cong 0,\\ \mathrm{Ext}^i_{\mathbf{D}_c(X)}(\mathcal{S}_2,\mathcal{S}_1) &\cong \mathrm{Ext}^i_{\mathbf{D}_c(X)}(j_!\Bbbk_{\mathbb{C}^2}[1], i_*\Bbbk_{\mathbb{P}^1}[1]) \cong 0,\\ \mathrm{Ext}^i_{\mathbf{D}_c(X)}(\mathcal{S}_2,\mathcal{S}_2) &\cong \mathrm{Ext}^i_{\mathbf{D}_c(X)}(j_!\Bbbk_{\mathbb{C}^2}[1], j_!\Bbbk_{\mathbb{C}^2}[1]) \cong H^i(\mathbb{C}^2;\Bbbk) \cong \begin{cases}\Bbbk & \text{if } i=0\\ 0 & \text{if } i\neq 0\end{cases}.\end{aligned}$$

Here, in order to compute $\mathrm{Ext}^i_{\mathbf{D}_c(X)}(\mathcal{E}, j_!\Bbbk_{\mathbb{C}^2}[1])$ for $\mathcal{E}\in\{i_*t_*\Bbbk_{\mathbb{P}^0}, i_*\Bbbk_{\mathbb{P}^1}[1]\}$ we applied the functor $\mathrm{Ext}^i_{\mathbf{D}_c(X)}(\mathcal{E},-)$ to the triangle $i^!j_!\Bbbk_{\mathbb{C}^2}[1]\to j_!\Bbbk_{\mathbb{C}^2}[1]\to j_*\Bbbk_{\mathbb{C}^2}[1]\to i^!j_!\Bbbk_{\mathbb{C}^2}[2]$ to get the long exact sequence

$$\ldots\to \mathrm{Ext}^i_{\mathbf{D}_c(X)}(\mathcal{E}, i^!j_!\Bbbk_{\mathbb{C}^2}[1]) \to \mathrm{Ext}^i_{\mathbf{D}_c(X)}(\mathcal{E}, j_!\Bbbk_{\mathbb{C}^2}[1]) \to \mathrm{Ext}^i_{\mathbf{D}_c(X)}(\mathcal{E}, j_*\Bbbk_{\mathbb{C}^2}[1]) \to \ldots$$

and note that $\mathrm{Ext}^i_{\mathbf{D}_c(X)}(\mathcal{E}, j_*\Bbbk_{\mathbb{C}^2}[1]) \cong \mathrm{Ext}^i_{\mathbf{D}_c(Z)}(\mathcal{E}|_Z, i^!j_*\Bbbk_{\mathbb{C}^2}[1]) \cong 0\ \forall i$. Therefore, we have

$$\mathrm{Ext}^i_{\mathbf{D}_c(X)}(\mathcal{E}, j_!\Bbbk_{C^2}[1]) \cong \mathrm{Ext}^i_{\mathbf{D}_c(Z)}(\mathcal{E}|_Z, i^!j_!\Bbbk_{\mathbb{C}^2}[1]) \cong \mathrm{Ext}^i_{\mathbf{D}_c(Z)}(\mathcal{E}|_Z, i^*j_*\Bbbk_{\mathbb{C}^2}).$$

Ext-Quiver and Relations

The Ext-quiver $\mathrm{Q}_q(\mathbb{P}^2)$ of the category ${}^q\mathbf{Perv}(X)$ is given by the Ext^1-groups between simple objects. Therefore, it is

$$
\begin{tikzcd}
2 & 1 \arrow[l, "\gamma"] \arrow[r, bend left, "\alpha"] & 0 \arrow[l, bend left, "\beta"]
\end{tikzcd}
.
$$

The ideal of relations $\mathrm{I}_q(\mathbb{P}^2)$ arises from Ext^2-groups between simple objects. We know that $\mathrm{I}_q(\mathbb{P}^2)$ is generated by at most two relations in $e_0\mathrm{I}_q(\mathbb{P}^2)e_2$ and $e_1\mathrm{I}_q(\mathbb{P}^2)e_1$. By the same argument used for ${}^m\mathbf{Perv}(\mathbb{P}^2)$ in Section 6.2.2, we have $\beta \circ \alpha \in \mathrm{I}_q(\mathbb{P}^2)$. Moreover, we know that there is a relation

$$\gamma \circ \beta + \{\text{paths of length} > 2\} \in e_0\mathrm{I}_m(\mathbb{P}^2)e_2,$$

but all the higher terms of length greater than two are already in $\mathrm{I}_q(\mathbb{P}^2)$ and hence $\gamma \circ \beta \in \mathrm{I}_q(\mathbb{P}^2)$. Therefore the relations are

$$\beta \circ \alpha = 0 \qquad \text{and} \qquad \gamma \circ \beta = 0,$$

which correspond to the (clockwise) cycle at the vertex 1 and the length two path from 0 to 2 respectively. Note that even though $\mathrm{Ext}^2_{\mathbf{D}_c(X)}(\mathcal{S}_1, \mathcal{S}_2) \neq 0$, it does not produce a relation as the only non-trivial path from 1 to 2 is the edge labelled by γ. Once again, note that the relations are quadratic.

Indecomposable Objects

Using the equivalence of categories ${}^q\mathbf{Perv}(\mathbb{P}^2) \simeq \mathbf{rep}(\mathrm{Q}_q(\mathbb{P}^2), \mathrm{I}_q(\mathbb{P}^2))$, we can explicitly list all the indecomposable perverse sheaves. They can equivalently characterised as irreducible representations of the Ext-quiver with relations $(\mathrm{Q}_q(X), \mathrm{I}_q(X))$ and diagrammatically, as a quiver with a map to $\mathrm{Q}_q(\mathbb{P}^2)$ with one vertex for each element of a basis of the representation and an arrow between vertices when the source basis element maps to the target one under the corresponding image arrow in $\mathrm{Q}_q(\mathbb{P}^2)$. We also give the corresponding dimension vector of each irreducible quiver representation.

Object	Quiver Representation	Path	Dimension Vector
$\mathcal{S}_0$	$0 \longleftarrow 0 \rightleftarrows \Bbbk$	$\times \quad \times \quad \bullet$	(001)
$\mathcal{S}_1$	$0 \longleftarrow \Bbbk \rightleftarrows 0$	$\times \quad \bullet \quad \times$	(010)
$\mathcal{S}_2 \cong \mathcal{P}_2$	$\Bbbk \longleftarrow 0 \rightleftarrows 0$	$\bullet \quad \times \quad \times$	(100)
$\widehat{\mathcal{P}}_1$	$0 \longleftarrow \Bbbk \underset{0}{\overset{1}{\rightleftarrows}} \Bbbk$	$\times \quad \bullet \rightarrow \bullet$	(011)
$\mathcal{I}_2$	$\Bbbk \xleftarrow{1} \Bbbk \rightleftarrows 0$	$\bullet \longleftarrow \bullet \quad \times$	(110)
$\mathcal{I}_1$	$0 \longleftarrow \Bbbk \underset{1}{\overset{0}{\rightleftarrows}} \Bbbk$	$\times \quad \bullet \leftarrow \bullet$	(011)
$\mathcal{P}_1$	$\Bbbk \xleftarrow{1} \Bbbk \underset{0}{\overset{1}{\rightleftarrows}} \Bbbk$	$\bullet \longleftarrow \bullet \rightarrow \bullet$	(111)
$\mathcal{P}_0 \cong \mathcal{I}_0$	$0 \longleftarrow \Bbbk \underset{(01)^t}{\overset{(10)}{\rightleftarrows}} \Bbbk^2$	$\times \quad \bullet \rightleftarrows {\bullet \atop \bullet}$	(012)

Table 6.2: Indecomposable perverse sheaves in ${}^q\mathbf{Perv}(\mathbb{P}^2)$.

Minimal Projective Presentations and Global Dimension

The minimal projective presentations of the three simple objects are respectively

$$\mathcal{P}_1 \to \mathcal{P}_0 \twoheadrightarrow \mathcal{S}_0,$$
$$\mathcal{P}_0 \oplus \mathcal{P}_2 \to \mathcal{P}_1 \twoheadrightarrow \mathcal{S}_1,$$
$$\mathcal{P}_2 \xrightarrow{\cong} \mathcal{S}_2.$$

One can extend the above minimal projective presentations of simple objects to get minimal projective resolutions of $\mathcal{S}_i \in {}^q\mathbf{Perv}(\mathbb{P}^2)$, which are

$$\mathcal{P}_0^\bullet = \mathcal{P}_2 \hookrightarrow \mathcal{P}_1 \to \mathcal{P}_0,$$
$$\mathcal{P}_1^\bullet = \mathcal{P}_2 \hookrightarrow \mathcal{P}_1 \to \mathcal{P}_0 \oplus \mathcal{P}_2 \to \mathcal{P}_1.$$

Thus, we have $\mathrm{gldim}({}^q\mathbf{Perv}(\mathbb{P}^2)) = 3$.

AR-quiver

The vertices of the Auslander-Reiten quiver of the category ${}^{q}\mathbf{Perv}(\mathbb{P}^2)$ are given by indecomposable perverse sheaves, therefore they are objects appearing in the list of Table 6.2. In the same way explained in Section 6.2.2, we can find the Auslander-Reiten quiver of ${}^{q}\mathbf{Perv}(\mathbb{P}^2)$ by adding the stratum S_2 to ${}^{m}\mathbf{Perv}(\mathbb{P}^1)$. In particular, there are three extensions of $\mathcal{S}_n$ supported on X, namely $\mathcal{P}_2 \hookrightarrow \mathcal{P}_1 \twoheadrightarrow \mathcal{I}_2$. One then needs to check, using Lemma 2.2.3.7, that the previous morphisms are irreducible. Since $\mathrm{rad}(\mathcal{P}_1) \cong \mathcal{S}_1 \oplus \mathcal{S}_0$, then $\mathcal{S}_0 \hookrightarrow \mathcal{P}_1$ is an irreducible morphism as well as $\mathcal{P}_1 \twoheadrightarrow \widehat{\mathcal{P}}_1$. Finally, the morphism $\mathcal{I}_2 \twoheadrightarrow \mathcal{S}_1$ is also irreducible. This gives the Auslander-Reiten quiver of the category ${}^{q}\mathbf{Perv}(\mathbb{P}^2)$.

$$
\begin{tikzcd}
= & \substack{\mathcal{S}_0\\(001)} \arrow[dr, hook] & & \substack{\mathcal{I}_2\\(110)} \arrow[ll, dashed] \arrow[dr, two heads] & & & & \\
& & \substack{\mathcal{P}_1\\(111)} \arrow[ur, two heads] \arrow[dr, two heads] & & \substack{\mathcal{S}_1\\(010)} \arrow[ll, dashed] \arrow[dr, hook] & & \substack{\mathcal{S}_0\\(001)} \arrow[ll, dashed] & = \\
& \substack{\mathcal{P}_2\cong\mathcal{S}_2\\(100)} \arrow[ur, hook] & & \substack{\widehat{\mathcal{P}}_1\\(011)} \arrow[ll, dashed] \arrow[ur, two heads] \arrow[dr, hook] & & \substack{\mathcal{I}_1\\(011)} \arrow[ll, dashed] \arrow[ur, two heads] & & \\
& & & & \substack{\mathcal{P}_0\cong\mathcal{I}_0\\(012)} \arrow[ur, two heads] & & &
\end{tikzcd}
$$

Figure 6.5: The Auslander-Reiten quiver of the category ${}^{q}\mathbf{Perv}(\mathbb{P}^2)$.

One can apply a similar argument to the one used in Section 6.2.2 in order to check that the maps in Figure 6.5 are all the irreducible maps.

On Faithfulness

The heart ${}^{q}\mathbf{Perv}(\mathbb{P}^2) \subset \mathbf{D}_c(\mathbb{P}^2)$ is not faithful. The object $\mathcal{I}_2 \in {}^{q}\mathbf{Perv}(\mathbb{P}^2)$ sits in the short exact sequence $0 \to \mathcal{S}_1 \to \mathcal{I}_2 \to \mathcal{S}_2 \to 0$, hence $\mathcal{I}_2 \cong \Bbbk_X[1]$. Then, we have

$$\mathrm{Ext}^i_{{}^{q}\mathbf{Perv}(\mathbb{P}^2)}(\mathcal{E}, \mathcal{I}_2) \cong 0 \quad \forall i \geq 1$$

for any $\mathcal{E} \in {}^{q}\mathbf{Perv}(X)$, since $\mathcal{I}_2$ is injective in ${}^{q}\mathbf{Perv}(\mathbb{P}^2)$. On the other hand,

$$\mathrm{Ext}^i_{\mathbf{D}_c(X)}(\mathcal{I}_2, \mathcal{I}_2) \cong H^i(X; \Bbbk) \cong \begin{cases} \Bbbk & \text{if } \ i = 0, 2, 4 \\ 0 & \text{otherwise} \end{cases}.$$

6.2.4 ${}^{s}\mathbf{Perv}(\mathbb{P}^2)$

Let us consider the perversity $s = (-1, 0, 0)$ obtained by subtracting $(0, 0, 1)$ from the middle perversity (and then shifted in order to have $s(S_2) = 0$).

Simple Objects

The three simple objects in ${}^{s}\mathbf{Perv}(\mathbb{P}^2)$ can be determined by using Proposition 5.2.2.3. They are (after shifting)

$$\mathcal{S}_0 = i_* t_* \Bbbk_{\mathbb{P}^0}, \qquad \mathcal{S}_1 = i_* l_! \Bbbk_{\mathbb{C}} \qquad \text{and} \qquad \mathcal{S}_2 = \Bbbk_X[1].$$

Alternatively, one can check that they satisfy the strong conditions of Remark 2.3.5.5.iii), therefore they are isomorphic to the simple objects. Note that the simple object supported on S_0 coincides with the one of ${}^{m}\mathbf{Perv}(\mathbb{P}^2)$, while the simple object supported on S_2 is a shift of the corresponding one in ${}^{m}\mathbf{Perv}(\mathbb{P}^2)$.

Ext-**Algebra**

We compute the Ext-groups $\mathrm{Ext}^k_{\mathbf{D}_c(X)}(\mathcal{S}_i, \mathcal{S}_j)$ for $i, j = 0, 1, 2$ in order to determine the Ext-algebra. We omit the calculation of $\mathrm{Ext}^i_{\mathbf{D}_c(X)}(\mathcal{S}_k, \mathcal{S}_k)$ for $k = 0, 2$ as it is the same as the one in Section 6.2.2.

$$\begin{aligned}
\mathrm{Ext}^i_{\mathbf{D}_c(X)}(\mathcal{S}_0, \mathcal{S}_1) &\cong \mathrm{Ext}^i_{\mathbf{D}_c(X)}(i_* t_* \Bbbk_{\mathbb{P}^0}, i_* l_! \Bbbk_{\mathbb{C}}) \cong \mathrm{Ext}^i_{\mathbf{D}_c(S_0)}(\Bbbk_{\mathbb{P}^0}, t^! l_! \Bbbk_{\mathbb{C}}) \\
&\cong \mathrm{Ext}^i_{\mathbf{D}_c(S_0)}(\Bbbk_{\mathbb{P}^0}, t^* l_* \Bbbk_{\mathbb{C}}[-1]) \cong \mathrm{Ext}^i_{\mathbf{D}_c(S_0)}(\Bbbk_{\mathbb{P}^0}, \Bbbk_{\mathbb{P}^0}[-1]) \cong \begin{cases} \Bbbk & \text{if } i = 1 \\ 0 & \text{if } i \neq 1 \end{cases}, \\
\mathrm{Ext}^i_{\mathbf{D}_c(X)}(\mathcal{S}_0, \mathcal{S}_2) &\cong \mathrm{Ext}^i_{\mathbf{D}_c(X)}(i_* t_* \Bbbk_{\mathbb{P}^0}, \Bbbk_X[1]) \cong \mathrm{Ext}^i_{\mathbf{D}_c(S_0)}(\Bbbk_{\mathbb{P}^0}, t^! i^! \Bbbk_X[1]) \\
&\cong \mathrm{Ext}^i_{\mathbf{D}_c(S_0)}(\Bbbk_{\mathbb{P}^0}, \Bbbk_{\mathbb{P}^0}[-3]) \cong \begin{cases} \Bbbk & \text{if } i = 3 \\ 0 & \text{if } i \neq 3 \end{cases},
\end{aligned}$$

$$
\begin{aligned}
\operatorname{Ext}^i_{\mathbf{D}_c(X)}(\mathcal{S}_1,\mathcal{S}_0) &\cong \operatorname{Ext}^i_{\mathbf{D}_c(X)}(i_*l_!\Bbbk_{\mathbb{C}}, i_*t_*\Bbbk_{\mathbb{P}^0}) \cong \operatorname{Ext}^i_{\mathbf{D}_c(Z)}(l_!\Bbbk_{\mathbb{C}}, t_*\Bbbk_{\mathbb{P}^0}) \cong 0,\\
\operatorname{Ext}^i_{\mathbf{D}_c(X)}(\mathcal{S}_1,\mathcal{S}_1) &\cong \operatorname{Ext}^i_{\mathbf{D}_c(X)}(i_*l_!\Bbbk_{\mathbb{C}}, i_*l_!\Bbbk_{\mathbb{C}}) \cong H^i(\mathbb{C};\Bbbk) \cong \begin{cases} \Bbbk & \text{if } i=0\\ 0 & \text{if } i\neq 0\end{cases},\\
\operatorname{Ext}^i_{\mathbf{D}_c(X)}(\mathcal{S}_1,\mathcal{S}_2) &\cong \operatorname{Ext}^i_{\mathbf{D}_c(X)}(i_*l_!\Bbbk_{\mathbb{C}}, \Bbbk_X[1]) \cong \operatorname{Ext}^i_{\mathbf{D}_c(Z)}(l_!\Bbbk_{\mathbb{C}}, i^!\Bbbk_X[1])\\
&\cong \operatorname{Ext}^i_{\mathbf{D}_c(Z)}(l_!\Bbbk_{\mathbb{C}}, \Bbbk_{\mathbb{P}^1}[-1]) \cong \begin{cases} \Bbbk & \text{if } i=1\\ 0 & \text{if } i\neq 1\end{cases},\\
\operatorname{Ext}^i_{\mathbf{D}_c(X)}(\mathcal{S}_2,\mathcal{S}_0) &\cong \operatorname{Ext}^i_{\mathbf{D}_c(X)}(\Bbbk_X[1], i_*t_*\Bbbk_{\mathbb{P}^0}) \cong \operatorname{Ext}^i_{\mathbf{D}_c(S_0)}(t^*i^*\Bbbk_X[1], \Bbbk_{\mathbb{P}^0})\\
&\cong \operatorname{Ext}^i_{\mathbf{D}_c(S_0)}(\Bbbk_{\mathbb{P}^0}[1], \Bbbk_{\mathbb{P}^0}) \cong \begin{cases} \Bbbk & \text{if } i=1\\ 0 & \text{if } i\neq 1\end{cases},\\
\operatorname{Ext}^i_{\mathbf{D}_c(X)}(\mathcal{S}_2,\mathcal{S}_1) &\cong \operatorname{Ext}^i_{\mathbf{D}_c(X)}(\Bbbk_X[1], i_*l_!\Bbbk_{\mathbb{C}}) \cong \operatorname{Ext}^i_{\mathbf{D}_c(Z)}(i^*\Bbbk_X[1], l_!\Bbbk_{\mathbb{C}})\\
&\cong \operatorname{Ext}^i_{\mathbf{D}_c(Z)}(\Bbbk_{\mathbb{P}^1}[1], l_!\Bbbk_C) \cong \begin{cases} \Bbbk & \text{if } i=3\\ 0 & \text{if } i\neq 3\end{cases}.
\end{aligned}
$$

Here, in order to compute $\operatorname{Ext}^i_{\mathbf{D}_c(X)}(\mathcal{S}_2,\mathcal{S}_1)$ we apply the functor $\operatorname{Ext}^i_{\mathbf{D}_c(Z)}(\Bbbk_{\mathbb{P}^1}[1],-)$ to the triangle $l_!\Bbbk_{\mathbb{C}} \to \Bbbk_{\mathbb{P}^1} \to t_*\Bbbk_{\mathbb{P}^0} \to l_!\Bbbk_{\mathbb{C}}[1]$ to get the long exact sequence

$$\ldots \to \operatorname{Ext}^i_{\mathbf{D}_c(Z)}(\Bbbk_{\mathbb{P}^1}[1], l_!\Bbbk_{\mathbb{C}}) \to \operatorname{Ext}^i_{\mathbf{D}_c(Z)}(\Bbbk_{\mathbb{P}^1}[1], \Bbbk_{\mathbb{P}^1}) \to \operatorname{Ext}^i_{\mathbf{D}_c(Z)}(\Bbbk_{\mathbb{P}^1}[1], t_*\Bbbk_{\mathbb{P}^0}) \to \ldots$$

and then use that

$$\operatorname{Ext}^i_{\mathbf{D}_c(Z)}(\Bbbk_{\mathbb{P}^1}[1], \Bbbk_{\mathbb{P}^1}) \cong \begin{cases} \Bbbk & \text{if } i=1,3\\ 0 & \text{otherwise}\end{cases} \qquad \operatorname{Ext}^i_{\mathbf{D}_c(Z)}(\Bbbk_{\mathbb{P}^1}[1], t_*\Bbbk_{\mathbb{P}^0}) \begin{cases} \Bbbk & \text{if } i=1\\ 0 & \text{if } i\neq 1\end{cases}.$$

Ext-**Quiver and Relations**

The Ext-quiver of the category ${}^s\mathbf{Perv}(X)$ can be constructed by considering the Ext^1-groups between simple objects. Therefore, $\mathrm{Q}_s(\mathbb{P}^2)$ is given by

$$\begin{tikzcd} 2 \arrow[rr, bend left, "\alpha"] & 1 \arrow[l, bend left, "\gamma"] & 0 \arrow[l, bend left, "\beta"] \end{tikzcd} .$$

The relations $\mathrm{I}_s(\mathbb{P}^2)$ are induced by Ext^2-groups between simple objects. We know that

$\mathrm{I}_s(\mathbb{P}^2)$ is generated by at most one term in $e_2\mathrm{I}_s(\mathbb{P}^2)e_2$ and $\gamma \circ \beta \circ \alpha \in \mathrm{I}_s(\mathbb{P}^2)$ by Lemma 4.2.3.8. Hence $\langle \gamma \circ \beta \circ \alpha \rangle \cong \mathrm{I}_s(\mathbb{P}^2)$. Note that the ideal of relations $\mathrm{I}_s(\mathbb{P}^2)$ is not quadratic.

Indecomposable Objects

Using the equivalence of categories ${}^s\mathbf{Perv}(\mathbb{P}^2) \simeq \mathbf{rep}(\mathrm{Q}_s(\mathbb{P}^2), \mathrm{I}_s(\mathbb{P}^2))$, we can count all the indecomposable perverse sheaves. They are equivalently characterised as irreducible representations of the Ext-quiver with relations $(\mathrm{Q}_s(X), \mathrm{I}_s(X))$ or diagrammatically, as a quiver with a map to $\mathrm{Q}_s(\mathbb{P}^2)$ with one vertex for each element of a basis of the representation and an arrow between vertices when the source basis element maps to the target one under the corresponding image arrow in $\mathrm{Q}_s(\mathbb{P}^2)$. We also give the corresponding dimension vector of each irreducible quiver representation.

Object	Quiver Representation	Path	Dimension Vector
$\mathcal{S}_0$	$\begin{tikzcd} 0 \arrow[rr, bend left] & 0 \arrow[l, bend left] & \Bbbk \arrow[l, bend left] \end{tikzcd}$	$\times \quad \times \quad \bullet$	(001)
$\mathcal{S}_1$	$\begin{tikzcd} 0 \arrow[rr, bend left] & \Bbbk \arrow[l, bend left] & 0 \arrow[l, bend left] \end{tikzcd}$	$\times \quad \bullet \quad \times$	(010)
$\mathcal{S}_2$	$\begin{tikzcd} \Bbbk \arrow[rr, bend left] & 0 \arrow[l, bend left] & 0 \arrow[l, bend left] \end{tikzcd}$	$\bullet \quad \times \quad \times$	(100)
$\mathcal{B}_2$	$\begin{tikzcd} \Bbbk \arrow[rr, bend left, "1"] & 0 \arrow[l, bend left] & \Bbbk \arrow[l, bend left] \end{tikzcd}$	$\begin{tikzcd} \bullet \arrow[rr, bend left] & \times & \bullet \end{tikzcd}$	(101)
$\widehat{\mathcal{I}}_1$	$\begin{tikzcd} 0 \arrow[rr, bend left] & \Bbbk \arrow[l, bend left] & \Bbbk \arrow[l, bend left, "1"] \end{tikzcd}$	$\begin{tikzcd} \times & \bullet & \bullet \arrow[l, bend left] \end{tikzcd}$	(011)
$\overline{\mathcal{I}}_2$	$\begin{tikzcd} \Bbbk \arrow[rr, bend left] & \Bbbk \arrow[l, bend left, "1"] & 0 \arrow[l, bend left] \end{tikzcd}$	$\begin{tikzcd} \bullet & \bullet \arrow[l, bend left] & \times \end{tikzcd}$	(110)
$\mathcal{I}_2$	$\begin{tikzcd} \Bbbk \arrow[rr, bend left, "0"] & \Bbbk \arrow[l, bend left, "1"] & \Bbbk \arrow[l, bend left, "1"] \end{tikzcd}$	$\begin{tikzcd} \bullet & \bullet \arrow[l, bend left] & \bullet \arrow[l, bend left] \end{tikzcd}$	(111)
$\mathcal{M}_2$	$\begin{tikzcd} \Bbbk \arrow[rr, bend left, "1"] & \Bbbk \arrow[l, bend left, "1"] & \Bbbk \arrow[l, bend left, "0"] \end{tikzcd}$	$\begin{tikzcd} \bullet \arrow[rr, bend left] & \bullet \arrow[l, bend left] & \bullet \end{tikzcd}$	(111)
$\mathcal{P}_2$	$\begin{tikzcd} \Bbbk \arrow[rr, bend left, "1"] & \Bbbk \arrow[l, bend left, "0"] & \Bbbk \arrow[l, bend left, "1"] \end{tikzcd}$	$\begin{tikzcd} \bullet \arrow[rr, bend left] & \bullet & \bullet \arrow[l, bend left] \end{tikzcd}$	(111)
$\mathcal{I}_0$	$\begin{tikzcd} \Bbbk \arrow[rr, bend left, "(01)"] & \Bbbk \arrow[l, bend left, "(10)^t"] & \Bbbk^2 \arrow[l, bend left, "(01)"] \end{tikzcd}$	$\begin{tikzcd} \bullet \arrow[rr, bend left] & \bullet \arrow[l, bend left] & \substack{\bullet \\ \bullet} \arrow[l, bend left] \end{tikzcd}$	(112)
$\mathcal{P}_1$	$\begin{tikzcd} \Bbbk \arrow[rr, bend left, "1"] & \Bbbk^2 \arrow[l, bend left, "(10)^t"] & \Bbbk \arrow[l, bend left, "(01)"] \end{tikzcd}$	$\begin{tikzcd} \bullet \arrow[rr, bend left] & \substack{\bullet \\ \bullet} \arrow[l, bend left] & \bullet \arrow[l, bend left] \end{tikzcd}$	(121)
$\mathcal{P}_0 \cong \mathcal{I}_1$	$\begin{tikzcd} \Bbbk \arrow[rr, bend left, "(01)"] & \Bbbk^2 \arrow[l, bend left, "(10)^t"] & \Bbbk^2 \arrow[l, bend left, "\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)"] \end{tikzcd}$	$\begin{tikzcd} \bullet \arrow[rr, bend left] & \substack{\bullet \\ \bullet} \arrow[l, bend left] & \substack{\bullet \\ \bullet} \arrow[l, bend left] \end{tikzcd}$	(122)

Table 6.3: Indecomposable perverse sheaves in ${}^s\mathbf{Perv}(\mathbb{P}^2)$.

Minimal Projective Presentations and Global Dimension

The minimal projective presentations of the three simple objects are respectively

$$
\begin{aligned}
&\mathcal{P}_1 \hookrightarrow \mathcal{P}_0 \twoheadrightarrow \mathcal{S}_0,\\
&\mathcal{P}_2 \hookrightarrow \mathcal{P}_1 \twoheadrightarrow \mathcal{S}_1,\\
&\mathcal{P}_0 \rightarrow \mathcal{P}_2 \twoheadrightarrow \mathcal{S}_2.
\end{aligned}
$$

One can extend the above minimal projective presentations of simple objects to get minimal projective resolutions of $\mathcal{S}_i \in {}^s\mathbf{Perv}(\mathbb{P}^2)$, that is

$$\begin{aligned}
&\mathcal{P}_1 \hookrightarrow \mathcal{P}_0,\\
&\mathcal{P}_2 \hookrightarrow \mathcal{P}_1,\\
&\mathcal{P}_2 \hookrightarrow \mathcal{P}_0 \to \mathcal{P}_2
\end{aligned}$$

Thus, we have $\mathrm{gldim}({}^s\mathbf{Perv}(\mathbb{P}^2)) = 2$.

AR-quiver

The vertices of the Auslander-Reiten quiver of the category ${}^s\mathbf{Perv}(\mathbb{P}^2)$ are given by indecomposable perverse sheaves, therefore they are objects appearing in the list of Table 6.3. In the same way descibed in Section 6.2.2, one can build the square relative to objects of ${}^s\mathbf{Perv}(\mathbb{P}^2)$ supported on S_2 and then connect it with ${}^o\mathbf{Perv}(\mathbb{P}^1)$ twice, one time on the left of the central square and one time on the right. Indeed, one can check that the morphisms $\widehat{\mathcal{I}}_1 \hookrightarrow \mathcal{P}_2$ and $\mathcal{S}_0 \hookrightarrow \mathcal{B}_2$ (and dually $\mathcal{I}_2 \twoheadrightarrow \widehat{\mathcal{I}}_1$ and $\widehat{\mathcal{I}}_2 \twoheadrightarrow \mathcal{S}_1$) are irreducible. Therefore we can draw the Auslander-Reiten quiver of the category ${}^s\mathbf{Perv}(\mathbb{P}^2)$ as follows.

$$\begin{tikzcd}
 & & & & \substack{\mathcal{P}_0\cong\mathcal{I}_1\\(112)} \arrow[dr, twoheadrightarrow] & & & & \\
 & & & \substack{\mathcal{P}_1\\(121)} \arrow[ur, hookrightarrow] \arrow[dr, twoheadrightarrow] & & \substack{\mathcal{I}_0\\(112)} \arrow[ll, dashed] \arrow[dr, twoheadrightarrow] & & & \\
 & & \substack{\mathcal{P}_2\\(111)} \arrow[ur, hookrightarrow] \arrow[dr, twoheadrightarrow] & & \substack{\mathcal{M}_2\\(111)} \arrow[ll, dashed] \arrow[ur, hookrightarrow] \arrow[dr, twoheadrightarrow] & & \substack{\mathcal{I}_2\\(111)} \arrow[ll, dashed] \arrow[dr, twoheadrightarrow] & & \\
 = & \substack{\widehat{\mathcal{I}}_1\\(011)} \arrow[ur, hookrightarrow] \arrow[dr, twoheadrightarrow] & & \substack{\mathcal{B}_2\\(101)} \arrow[ll, dashed] \arrow[ur, hookrightarrow] \arrow[dr, twoheadrightarrow] & & \substack{\widehat{\mathcal{I}}_2\\(110)} \arrow[ll, dashed] \arrow[ur, hookrightarrow] \arrow[dr, twoheadrightarrow] & & \substack{\widehat{\mathcal{I}}_1\\(011)} \arrow[ll, dashed] \arrow[dr, twoheadrightarrow] & = \\
 \substack{\mathcal{S}_1\\(010)} \arrow[ur, hookrightarrow] & & \substack{\mathcal{S}_0\\(001)} \arrow[ll, dashed] \arrow[ur, hookrightarrow] & & \substack{\mathcal{S}_2\\(100)} \arrow[ll, dashed] \arrow[ur, hookrightarrow] & & \substack{\mathcal{S}_1\\(010)} \arrow[ll, dashed] \arrow[ur, hookrightarrow] & & \substack{\mathcal{S}_0\\(001)} \arrow[ll, dashed]
\end{tikzcd}$$

Figure 6.6: The Auslander-Reiten quiver of the category ${}^s\mathbf{Perv}(\mathbb{P}^2)$.

On Faithfulness

The heart ${}^s\mathbf{Perv}(\mathbb{P}^2) \subset \mathbf{D}_{\mathrm{c}}(\mathbb{P}^2)$ is not faithful. One can note that $s' = (-1,-1,-1)$ is a nearby perversity for s with heart ${}^{o}\mathbf{Perv}(\mathbb{P}^2)[1]$. By Proposition 5.2.3.1, we know that for the open stratum S_2 we have ${}^{s}\mathcal{P}_2 \cong {}^{s'}\mathcal{P}_2$ if ${}^{s}\mathbf{Perv}(\mathbb{P}^2)$ is faithful. There is a triangle

$$\Bbbk_{\mathbb{P}^0} \to {}^{s}\mathcal{P}_2 \to \Bbbk_X[1] \to \Bbbk_{\mathbb{P}^0}[1]$$

in $\mathbf{D}_c(X)$, so that ${}^{s}\mathcal{P}_2 \cong \Bbbk_{X \smallsetminus \mathbb{P}^0}[1]$. However, ${}^{s'}\mathcal{P}_2 \cong \Bbbk_{\mathbb{C}^2}[1]$ and ${}^{s}\mathcal{P}_2 \ncong {}^{s'}\mathcal{P}_2$. Thus ${}^{s}\mathbf{Perv}(\mathbb{P}^2)$ is not faithful.

Remark 6.2.4.1. *The fact that* $\mathrm{Ext}^2_{\mathbf{D}_c(X)}(\mathcal{S}_2, \mathcal{S}_2) \neq 0$*, implies that there is a relation starting and ending at the vertex* 2 *of the* Ext*-quiver* Q_s*. Since, as noted before, the only possibility is that such relation is* $\gamma \circ \beta \circ \alpha = 0$*, this implies that the* Ext*-algebra is an example of a non quadratic algebra.*

6.2.5 ${}^{o}\mathbf{Perv}(\mathbb{P}^2) \simeq \mathbf{Constr}(\mathbb{P}^2)$

Let us consider the zero perversity, that is $p(S_i) = 0$ for $i = 0, 1, 2$.

Simple Objects

The simple objects in ${}^{o}\mathbf{Perv}(\mathbb{P}^2)$ are extensions by zero of constant sheaves on each stratum, that is, using the maps in (6.2), they are

$$\mathcal{S}_0 \cong i_* t_* \Bbbk_{\mathbb{P}^0}, \qquad \mathcal{S}_1 \cong i_* l_! \Bbbk_{\mathbb{C}} \qquad \text{and} \qquad \mathcal{S}_2 \cong j_! \Bbbk_{\mathbb{C}^2}.$$

Ext**-Algebra**

We compute the Ext-groups $\mathrm{Ext}^k_{\mathbf{D}_c(X)}(\mathcal{S}_i, \mathcal{S}_j)$ for $i, j = 0, 1, 2$ in order to determine the Ext-algebra. Note that, the Ext-groups between $\mathcal{S}_0$ and $\mathcal{S}_1$ and vice-versa coincide with the ones in Section 6.2.4, as the simple objects $\mathcal{S}_0$ and $\mathcal{S}_1$ are the same in ${}^{s}\mathbf{Perv}(\mathbb{P}^2)$ and ${}^{o}\mathbf{Perv}(\mathbb{P}^2)$, therefore we only compute the Ext^k-groups when $\mathcal{S}_2$ is involved. As noted in Section 6.2.3, we have

$$i^! j_! \Bbbk_{\mathbb{C}^2} \cong i^* j_* \Bbbk_{\mathbb{C}^2}[-1] \cong \Bbbk_{\mathbb{P}^1}[-1] \oplus \Bbbk_{\mathbb{P}^1}[-2].$$

Therefore, the Ext-algebra is as follows.

$$\operatorname{Ext}^i_{\mathbf{D}_c(X)}(\mathcal{S}_0,\mathcal{S}_2) \cong \operatorname{Ext}^i_{\mathbf{D}_c(X)}(i_*t_*\Bbbk_{\mathbb{P}^0}, j_!\Bbbk_{\mathbb{C}^2}) \cong \operatorname{Ext}^i_{\mathbf{D}_c(Z)}(t_*\Bbbk_{\mathbb{P}^0}, i^!j_!\Bbbk_{\mathbb{C}^2}) \cong \begin{cases} \Bbbk & \text{if } i=3,4 \\ 0 & \text{otherwise} \end{cases},$$

$$\operatorname{Ext}^i_{\mathbf{D}_c(X)}(\mathcal{S}_1,\mathcal{S}_2) \cong \operatorname{Ext}^i_{\mathbf{D}_c(X)}(i_*l_!\Bbbk_{\mathbb{C}}, j_!\Bbbk_{\mathbb{C}^2}) \cong \operatorname{Ext}^i_{\mathbf{D}_c(Z)}(l_!\Bbbk_{\mathbb{C}}, i^!j_!\Bbbk_{\mathbb{C}^2}) \cong \begin{cases} \Bbbk & \text{if } i=1,2 \\ 0 & \text{otherwise} \end{cases},$$

$$\operatorname{Ext}^i_{\mathbf{D}_c(X)}(\mathcal{S}_2,\mathcal{S}_0) \cong \operatorname{Ext}^i_{\mathbf{D}_c(X)}(j_!\Bbbk_{\mathbb{C}^2}, i_*t_*\Bbbk_{\mathbb{P}^0}) \cong 0,$$

$$\operatorname{Ext}^i_{\mathbf{D}_c(X)}(\mathcal{S}_2,\mathcal{S}_1) \cong \operatorname{Ext}^i_{\mathbf{D}_c(X)}(j_!\Bbbk_{\mathbb{C}^2}, i_*l_!\Bbbk_{\mathbb{C}}) \cong 0,$$

$$\operatorname{Ext}^i_{\mathbf{D}_c(X)}(\mathcal{S}_2,\mathcal{S}_2) \cong \operatorname{Ext}^i_{\mathbf{D}_c(X)}(j_!\Bbbk_{\mathbb{C}^2}, j_!\Bbbk_{\mathbb{C}^2}) \cong H^i(\mathbb{C}^2;\Bbbk) \cong \begin{cases} \Bbbk & \text{if } i=0 \\ 0 & \text{otherwise} \end{cases}.$$

Ext-**Quiver and Relations**

The Ext-quiver of the category ${}^o\mathbf{Perv}(X)$ can be constructed by considering the Ext^1-groups between simple objects. Therefore, $\mathrm{Q}_o(\mathbb{P}^2)$ is given by

$$2 \xleftarrow{\alpha} 1 \xleftarrow{\beta} 0.$$

Note that $\operatorname{Ext}^2_{\mathbf{D}_c(X)}(\mathcal{S}_1,\mathcal{S}_2) \neq 0$, but it does not produce a relation. Since all the other Ext^2-groups vanish, there are no relations, that is $\mathrm{I}_o(\mathbb{P}^2) = 0$.

Remark 6.2.5.1. *Note that the* Ext*-quiver* $\mathrm{Q}_o(\mathbb{P}^2)$ *of the category* ${}^o\mathbf{Perv}(\mathbb{P}^2) \simeq \mathbf{Constr}(\mathbb{P}^2)$ *is isomorphic to a quiver of type* $\mathbb{A}_3$*, see [Sch14, Section 3.1] (although our labelling of the vertices does not match the one in the reference).*

Indecomposable Objects

Using the equivalence of categories ${}^o\mathbf{Perv}(\mathbb{P}^2) \simeq \mathbf{rep}(\mathrm{Q}_s(\mathbb{P}^2))$, we can count all the indecomposable perverse sheaves. They can equivalently characterised as irreducible representations of the Ext-quiver $\mathrm{Q}_o(X)$ or diagrammatically, as a quiver with a map to $\mathrm{Q}_o(\mathbb{P}^2)$ with one vertex for each element of a basis of the representation and an arrow between vertices when the source basis element maps to the target one under the corresponding image arrow in $\mathrm{Q}_o(\mathbb{P}^2)$. The Ext-quiver $\mathrm{Q}_o(\mathbb{P}^2)$ is a quiver of type $\mathbb{A}_3$, hence the Auslander-Reiten quiver is well-known. We compute it anyway, using our methods, as a check.

Object	Quiver Representation	Path	Dimension Vector
$\mathcal{S}_0 \cong \mathcal{I}_0$	$0 \longleftarrow 0 \longleftarrow \Bbbk$	$\times \quad \times \quad \bullet$	(001)
$\mathcal{S}_1$	$0 \longleftarrow \Bbbk \longleftarrow 0$	$\times \quad \bullet \quad \times$	(010)
$\mathcal{S}_2 \cong \mathcal{P}_2$	$\Bbbk \longleftarrow 0 \longleftarrow 0$	$\bullet \quad \times \quad \times$	(100)
$\mathcal{I}_1$	$0 \longleftarrow \Bbbk \xleftarrow{1} \Bbbk$	$\times \quad \bullet \longleftarrow \bullet$	(011)
$\mathcal{P}_1$	$\Bbbk \longleftarrow \Bbbk \longleftarrow 0$	$\bullet \longleftarrow \bullet \quad \times$	(110)
$\mathcal{P}_0 \cong \mathcal{I}_2$	$\Bbbk \longleftarrow \Bbbk \longleftarrow \Bbbk$	$\bullet \longleftarrow \bullet \longleftarrow \bullet$	(111)

Table 6.4: Indecomposable perverse sheaves in ${}^{o}\mathbf{Perv}(\mathbb{P}^2) \simeq \mathbf{Constr}(\mathbb{P}^2)$.

Minimal Projective Presentations and Global Dimension

The minimal projective presentations of the three simple objects are respectively

$$\mathcal{P}_1 \hookrightarrow \mathcal{P}_0 \twoheadrightarrow \mathcal{S}_0$$
$$\mathcal{P}_2 \hookrightarrow \mathcal{P}_1 \twoheadrightarrow \mathcal{S}_1$$
$$\mathcal{P}_2 \xrightarrow{\cong} \mathcal{S}_2.$$

One can note that the first part of the minimal projective presentations of simple objects gives minimal projective resolutions of $\mathcal{S}_i \in {}^{s}\mathbf{Perv}(\mathbb{P}^2)$.

Therefore, we have $\mathrm{gldim}({}^{s}\mathbf{Perv}(\mathbb{P}^2)) = 1$.

AR-quiver

The vertices of the Auslander-Reiten quiver of the category ${}^{s}\mathbf{Perv}(\mathbb{P}^2)$ are given by indecomposable perverse sheaves, therefore they are objects appearing in the list of Table 6.4. Irreducible morphisms, that is edges in the Auslander-Reiten quiver, can be determined by using the techniques of Section 2.2.4. Note that, since in this case there are no relations, in order to determine the irreducible morphisms (and hence the Auslander-Reiten quiver) one can use the knitting algorithm, see [Sch14, Section 3.1]. Alternatively, one cand find the objects in ${}^{o}\mathbf{Perv}(\mathbb{P}^2)$ which are supported on S_2, namely $\mathcal{P}_2 \hookrightarrow \mathcal{P}_1 \hookrightarrow \mathcal{P}_0$, and connect the above three objects with ${}^{o}\mathbf{Perv}(\mathbb{P}^1)$. Indeed, the morphisms between projective covers of simple objects are irreducible, as well as $\mathcal{P}_0 \twoheadrightarrow \mathcal{I}_1$ and $\mathcal{P}_1 \twoheadrightarrow \mathcal{S}_1$. Therefore, the Auslander-Reiten quiver of the category ${}^{o}\mathbf{Perv}(\mathbb{P}^2) \simeq \mathbf{Constr}(\mathbb{P}^2)$ is as follows.

$$\begin{tikzcd}
 & & \substack{\mathcal{P}_0\cong\mathcal{I}_2 \\ (111)} \arrow[dr, two heads] & & \\
 & \substack{\mathcal{P}_1 \\ (110)} \arrow[ur, hook] \arrow[dr, two heads] & & \substack{\mathcal{I}_1 \\ (011)} \arrow[ll, dashed] \arrow[dr, two heads] & \\
\substack{\mathcal{P}_2\cong\mathcal{S}_2 \\ (100)} \arrow[ur, hook] & & \substack{\mathcal{S}_1 \\ (010)} \arrow[ll, dashed] \arrow[ur, hook] & & \substack{\mathcal{S}_0\cong\mathcal{I}_0 \\ (001)} \arrow[ll, dashed]
\end{tikzcd}$$

Figure 6.7: The Auslander-Reiten quiver of the category ${}^{o}\mathbf{Perv}(\mathbb{P}^2) \simeq \mathbf{Constr}(\mathbb{P}^2)$.

On Faithfulness

The heart ${}^{o}\mathbf{Perv}(\mathbb{P}^2) \subset \mathbf{D}_{\mathrm{c}}(\mathbb{P}^2)$ is not a faithful as the object $\mathcal{P}_0 \cong \mathcal{I}_2 \cong \Bbbk_X \in {}^{o}\mathbf{Perv}(\mathbb{P}^2)$ is such that

$$\mathrm{Ext}^i_{\mathbf{D}_{\mathrm{c}}(X)}(\mathcal{P}_0, \mathcal{P}_0) \cong H^i(X; \Bbbk) \cong \begin{cases} \Bbbk & i = 0, 2, 4 \\ 0 & \text{otherwise} \end{cases},$$

while

$$\mathrm{Ext}^i_{{}^{o}\mathbf{Perv}(\mathbb{P}^2)}(\mathcal{P}_0, \mathcal{E}) \cong 0 \quad \forall i \geq 1$$

for any $\mathcal{E} \in {}^{o}\mathbf{Perv}(\mathbb{P}^2)$ as $\mathcal{P}_0 \in {}^{o}\mathbf{Perv}(\mathbb{P}^2)$ is projective.

6.2.6 ${}^{p}\mathbf{Perv}(\mathbb{P}^2)$ for the other GM-perversities

As noted in Remark 6.2.1.1, (r, s), (q, u) and (o, t) are pairs of dual perversities. Therefore, we have

$${}^{t}\mathbf{Perv}(\mathbb{P}^2) \simeq \mathcal{D}({}^{o}\mathbf{Perv}(\mathbb{P}^2)), \quad {}^{r}\mathbf{Perv}(\mathbb{P}^2) \simeq \mathcal{D}({}^{s}\mathbf{Perv}(\mathbb{P}^2)) \quad \text{and} \quad {}^{u}\mathbf{Perv}(\mathbb{P}^2) \simeq \mathcal{D}({}^{q}\mathbf{Perv}(\mathbb{P}^2)).$$

Note that this completely describes the categories of p-perverse sheaves on $X = \mathbb{P}^2$ with the affine stratification for all possible choices of GM-perversities p on X.

6.3 Projective Space

In this section, we study perverse sheaves on $X = \mathbb{P}^n$ stratified as in Example 2.3.1.6.iii), that is with stratification induced by the affine filtration

$$X = \mathbb{P}^n \supset X_{n-1} = \mathbb{P}^{n-1} \supset \ldots \supset X_0 = \mathbb{P}^0 \supset \emptyset.$$

There are $n+1$ strata, given by $S_i = \mathbb{P}^i \smallsetminus \mathbb{P}^{i-1} \cong \mathbb{C}^i$ for any $i = 0, \ldots, n$. We will use induction by adding the stratum S_n to $X_{n-1} \cong \mathbb{P}^{n-1}$ in order to generalise some results of Section 6.2. That is, we will consider the complementary maps

$$S_n \overset{j}{\hookrightarrow} X \overset{i}{\hookleftarrow} Z = \bigcup_{i=0}^{n-1} S_i.$$

We will focus our attention on two specific cases, namely the zero and the middle perversity.

6.3.1 ${}^{o}\mathbf{Perv}(\mathbb{P}^n)$ and ${}^{t}\mathbf{Perv}(\mathbb{P}^n)$

Let us consider the zero perversity, that is the perversity given by $o(S_i) = 0$ for any $i = 0, \ldots, n$.

Simple Objects

Let $i_S : S_i \hookrightarrow X$ be the inclusion of a stratum into $X = \mathbb{P}^n$. There is one simple objects for each stratum given by the extension by zero of the constant sheaf on the considered stratum, that is $\mathcal{S}_i \cong i_{S!}\Bbbk_{\mathbb{C}^i}$ for $i = 0, \ldots, n$.

Ext-**Algebra**

By inductive hypothesis on the number of strata, we can consider understood the Ext-algebra of ${}^{m}\mathbf{Perv}(\mathbb{P}^{n-1})$ and we can add the open stratum S_n. Therefore, the Ext-groups $\mathrm{Ext}^k_{\mathbf{D}_c(\mathbb{P}^{n-1})}(\mathcal{S}_i, \mathcal{S}_j)$ are known for $0 \le i, j \le n-1$ and we need to compute all the Ext-groups involving $\mathcal{S}_n$. Let $0 \le l \le n-1$, we have:

$$\begin{aligned}
&\mathrm{Ext}^k_{\mathbf{D}_c(X)}(\mathcal{S}_n,\mathcal{S}_n) \cong H^k(\mathbb{C}^n;\Bbbk) \cong \begin{cases} \Bbbk & \text{if } k=0 \\ 0 & \text{otherwise} \end{cases}, \\
&\mathrm{Ext}^k_{\mathbf{D}_c(X)}(\mathcal{S}_n,\mathcal{S}_l) \cong 0 \quad \text{for any } 0 \le l \le n-1 \quad \text{(by adjunction)}, \\
&\mathrm{Ext}^k_{\mathbf{D}_c(X)}(\mathcal{S}_l,\mathcal{S}_n) \cong \mathrm{Ext}^k_{\mathbf{D}_c(X)}(i_*\mathcal{S}_l,\mathcal{S}_n) \cong \mathrm{Ext}^k_{\mathbf{D}_c(\mathbb{P}^{n-1})}(\mathcal{S}_l,i^!\mathcal{S}_n) \\
&\qquad\qquad \cong \begin{cases} \Bbbk & \text{if } k=1 \text{ and } l=n-1 \\ 0 & \text{otherwise} \end{cases}.
\end{aligned} \tag{6.7}$$

Ext-Quiver and Relations

We can use the computations in (6.7) to (inductively) obtain the Ext-quiver $\mathrm{Q}_o(\mathbb{P}^n)$ of ${}^o\mathbf{Perv}(\mathbb{P}^n) \simeq \mathbf{Constr}(\mathbb{P}^n)$. We have

$$n \longleftarrow n-1 \longleftarrow \ldots \longleftarrow 1 \longleftarrow 0$$

There are no relations as all the Ext^2-groups vanish, that is $\mathrm{I}_o(\mathbb{P}^n)=0$. The Ext-quiver $\mathrm{Q}_o(\mathbb{P}^n)$ is a Dynkin quiver of type $\mathbb{A}_{n+1}$, therefore the indecomposable representations and the Auslander-Reiten quiver are well-known. For completeness, we construct it using our methods.

Indecomposable Objects

In order to count the indecomposable perverse sheaves in ${}^m\mathbf{Perv}(\mathbb{P}^n)$, one can find the indecomposable objects in the path algebra $\Bbbk\mathrm{Q}_o(\mathbb{P}^n)$, that this the path algebra of a quiver of type $\mathbb{A}_{n+1}$. Therefore we have the following.

Theorem 6.3.1.1. *In ${}^o\mathbf{Perv}(\mathbb{P}^n) \simeq \mathbf{Constr}(\mathbb{P}^n)$ there are $\displaystyle\sum_{i=1}^{n+1} i$ (isomorphism classes of) indecomposable objects.*

Proof. We proceed by induction by adding the vertex labelled by n to the Ext-quiver $\mathrm{Q}_o(\mathbb{P}^{n-1})$. By inductive hypothesis we have that there are $\sum_{i=1}^n i$ indecomposable objects in ${}^o\mathbf{Perv}(\mathbb{P}^{n-1})$ and we have to find the number of new indecomposable objects in ${}^o\mathbf{Perv}(\mathbb{P}^n)$ with support that contains S_n. Those are, the simple-projective object $\mathcal{S}_n \in {}^o\mathbf{Perv}(\mathbb{P}^n)$ and all the other indecomposable projective objects $\mathcal{P}_i$ for $i=0,\ldots,n-1$.

By considering the dimension vector of theese objects, which is well-defined as $\mathrm{I}_o(\mathbb{P}^n) = 0$, the above are all the new indecomposable objects with support containing S_n. Therefore, we have that the number of indecomposable objects in ${}^o\mathbf{Perv}(\mathbb{P}^n)$ is

$$\sum_{i=1}^{n} i + n + 1 = \sum_{i=1}^{n+1} i.$$

$\square$

Minimal Projective Resolutions and Global Dimension

We now consider minimal projective presentations of simple objects $\mathcal{S}_i \in {}^o\mathbf{Perv}(\mathbb{P}^n)$, that is exact sequences of the form

$$\mathcal{P}(\ker \pi_i) \to \mathcal{P}_i \overset{\pi_i}{\twoheadrightarrow} \mathcal{S}_i,$$

where $\mathcal{P}_i \in {}^o\mathbf{Perv}(X)$ is the projective cover of $\mathcal{S}_i$, π_i is the projective cover map and $\mathcal{P}(\mathcal{E})$ is the projective cover of the object $\mathcal{E} \in {}^o\mathbf{Perv}(X)$. We can note that

$$\ker(\pi_i) \cong \mathrm{rad}(\mathcal{P}_i) \cong \begin{cases} 0 & \text{if } i = 0 \\ \mathcal{P}_{i-1} & \text{if } 1 \leq i \leq n \end{cases}. \tag{6.8}$$

Therefore, the minimal projective presentations of simple objects are

$$\begin{aligned} &\mathcal{P}_{i-1} \hookrightarrow \mathcal{P}_i \twoheadrightarrow \mathcal{S}_i \quad \text{if } 0 \leq i \leq n-1 \\ &\mathcal{P}_n \overset{\cong}{\to} \mathcal{S}_n. \end{aligned}$$

The minimal projective resolutions are given by considering the above sequences without the last term, therefore we have $\mathrm{gldim}({}^o\mathbf{Perv}(\mathbb{P}^n)) = 1$.

AR-Quiver

One can determine the Auslander-Reiten quiver of ${}^o\mathbf{Perv}(\mathbb{P}^n) \simeq \mathbf{Constr}(\mathbb{P}^n)$ by using the knitting algorithm, as in this case the Ext-quiver has no relations. Alternatively, one can do it inductively by adding the stratum S_n to $\mathbb{P}^{n-1}$. It is then enough to determine the chain of extensions of objects supported on S_n,

$$\mathcal{S}_n \cong \mathcal{P}_n \hookrightarrow \mathcal{P}_{n-1} \hookrightarrow \ldots \hookrightarrow \mathcal{P}_1 \hookrightarrow \mathcal{P}_0 \cong \mathcal{I}_n,$$

and connect it to the Auslander-Reiten quiver of $\mathbb{P}^{n-1}$. In order to do so, it is enough to note that all the morphisms involved in the short exact sequences

$$\begin{aligned} &\mathcal{P}_i \hookrightarrow \mathcal{P}_{i-1} \twoheadrightarrow \widehat{\mathcal{P}}_{i-1} \quad \text{if } 2 \leq i \leq n \\ &\mathcal{P}_1 \hookrightarrow \mathcal{P}_0 \twoheadrightarrow \mathcal{I}_{n-1} \end{aligned}$$

are irreducible. Therefore, the Auslander-Reiten quiver of the category ${}^{o}\mathbf{Perv}(\mathbb{P}^n) \simeq \mathbf{Constr}(\mathbb{P}^n)$ is the following.

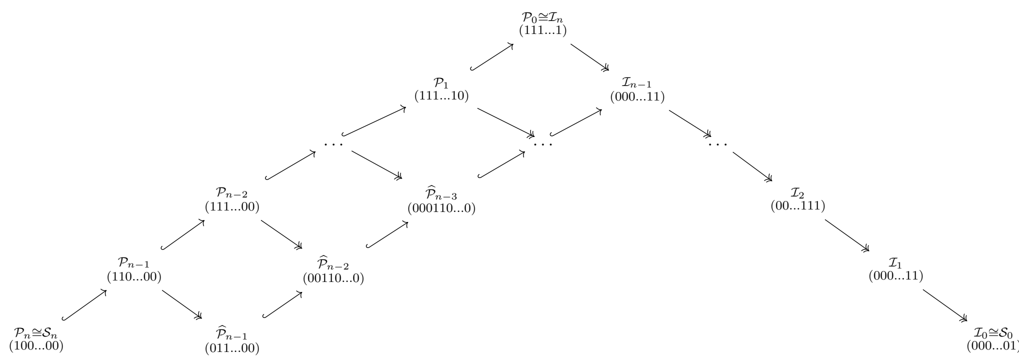

Figure 6.8: The Auslander-Reiten quiver of the category ${}^{o}\mathbf{Perv}(\mathbb{P}^n) \simeq \mathbf{Constr}(\mathbb{P}^n)$.

On Faithfulness

The heart ${}^{o}\mathbf{Perv}(\mathbb{P}^n)$ is not faithful as the object $\mathcal{P}_0 \cong \mathcal{I}_n \cong \Bbbk_X \in {}^{o}\mathbf{Perv}(\mathbb{P}^n)$ is such that

$$\mathrm{Ext}^i_{\mathbf{D}_c(X)}(\mathcal{P}_0, \mathcal{P}_0) \cong H^i(X; \Bbbk) \cong \begin{cases} \Bbbk & 0 \leq i \leq n \ \text{ i even} \\ 0 & \text{otherwise} \end{cases},$$

while

$$\mathrm{Ext}^i_{{}^{o}\mathbf{Perv}(\mathbb{P}^n)}(\mathcal{P}_0, \mathcal{E}) \cong 0$$

for any $\mathcal{E} \in {}^{o}\mathbf{Perv}(\mathbb{P}^n)$ as $\mathcal{P}_0 \in {}^{o}\mathbf{Perv}(\mathbb{P}^n)$ is projective.

6.3.2 ${}^{m}\mathbf{Perv}(\mathbb{P}^n)$

Let us consider the middle perversity, that is the perversity given by

$$m(S_i) = -\dim_{\mathbb{C}} S_i = -i$$

for any $i = 0, \dots, n$.

Simple Objects

There is one simple object $\mathcal{S}_i \in {}^{m}\mathbf{Perv}(\mathbb{P}^n)$ for each stratum. Since the closure of each stratum S_i is a smooth manifold, the simple objects are (extensions by zero of) shifted constant sheaves of the form $\Bbbk_{\mathbb{P}^i}[i]$ for $i = 0, \dots, n$.

Ext-**Algebra**

By inductive hypothesis on the number of strata, we can consider understood the Ext-algebra of ${}^{m}\mathbf{Perv}(\mathbb{P}^{n-1})$ and we can add the open stratum S_n. (Note that we could add the closed stratum S_0 instead, but this choice is less convenient). Therefore, the Ext-groups $\mathrm{Ext}^k_{\mathbf{D}_c(\mathbb{P}^{n-1})}(\mathcal{S}_i, \mathcal{S}_j)$ are known for $0 \leq i, j \leq n-1$ and we need to compute all the Ext-groups involving $\mathcal{S}_n$. Let $0 \leq l \leq n-1$, we have:

$$\begin{aligned}
\mathrm{Ext}^k_{\mathbf{D}_c(X)}(\mathcal{S}_n, \mathcal{S}_n) &\cong H^k(X; \Bbbk) \cong \begin{cases} \Bbbk & \text{if } 0 \leq i \leq n, \text{ i even} \\ 0 & \text{otherwise} \end{cases}, \\
\mathrm{Ext}^k_{\mathbf{D}_c(X)}(\mathcal{S}_n, \mathcal{S}_l) &\cong \mathrm{Ext}^k_{\mathbf{D}_c(Z)}(i^*\mathcal{S}_n, \mathcal{S}_l) \cong \mathrm{Ext}^k_{\mathbf{D}_c(Z)}(\mathcal{S}_{n-1}[1], \mathcal{S}_l), \\
\mathrm{Ext}^k_{\mathbf{D}_c(X)}(\mathcal{S}_l, \mathcal{S}_n) &\cong \mathrm{Ext}^k_{\mathbf{D}_c(Z)}(\mathcal{S}_l, i^!\mathcal{S}_n) \cong \mathrm{Ext}^k_{\mathbf{D}_c(Z)}(\mathcal{S}_l, \mathcal{S}_{n-1}[-1]).
\end{aligned}$$

Thus, in particular we have

$$\begin{aligned}
&\mathrm{Ext}^2_{\mathbf{D}_c(X)}(\mathcal{S}_n, \mathcal{S}_n) \cong \Bbbk, \\
&\mathrm{Ext}^i_{\mathbf{D}_c(X)}(\mathcal{S}_n, \mathcal{S}_l) \cong \mathrm{Ext}^i_{\mathbf{D}_c(X)}(\mathcal{S}_l, \mathcal{S}_n) \cong \begin{cases} \Bbbk & \text{if } i = 1,\ l = n-1 \\ \Bbbk & \text{if } i = 2,\ l = n-2 \\ 0 & \text{otherwise} \end{cases}.
\end{aligned} \tag{6.9}$$

Ext-**Quiver and Relations**

We can use the computations in (6.9) to (inductively) get the Ext-quiver $\mathrm{Q}_m(\mathbb{P}^n)$. We have

$$\begin{tikzcd}
n \arrow[r, bend left, "\alpha_n"] & n-1 \arrow[l, bend left, "\beta_n"] \arrow[r, bend left, "\alpha_{n-1}"] & \dots \arrow[l, bend left, "\beta_{n-1}"] \arrow[r, bend left, "\alpha_2"] & 1 \arrow[l, bend left, "\beta_2"] \arrow[r, bend left, "\alpha_1"] & 0 \arrow[l, bend left, "\beta_1"]
\end{tikzcd}$$

The relations $\mathrm{I}_m(\mathbb{P}^n)$ are the same as in $\mathrm{I}_m(\mathbb{P}^{n-1})$ with the addition of (potentially) three new relations involving the new vertex of $\mathrm{Q}_m(\mathbb{P}^n)$ labelled by n. By Lemma 4.2.3.8 we know that $\beta_n \circ \alpha_n \in e_n \mathrm{I}_m(\mathbb{P}^n) e_n$. By (4.5) and Lemma 4.2.3.2 we know that

$$\alpha_{n-1} \circ \alpha_n + \{\text{paths of length } {>}2\} \in e_n \mathrm{I}_m(\mathbb{P}^n) e_{n-2}.$$

However, all paths from 2 to 0 of length greater than two are already in $\mathrm{I}_m(\mathbb{P}^n)$, therefore $\alpha_{n-1} \circ \alpha_n \in \mathrm{I}_m(\mathbb{P}^n)$. A similar argument shows that $\beta_n \circ \beta_{n-1} \in \mathrm{I}_m(\mathbb{P}^n)$. That is, we add to $\mathrm{I}_m(\mathbb{P}^{n-1})$

$$\beta_n \circ \alpha_n = 0, \qquad \alpha_{n-1} \circ \alpha_n = 0 \qquad \text{and} \qquad \beta_n \circ \beta_{n-1} = 0,$$

where the first one represents the clockwise cycle around the vertex n, the second the length two path from n to $n-2$ and the latter the length two path from $n-2$ to n. Therefore, we have

$$\mathrm{I}_m(\mathbb{P}^n) = \{\beta_i \circ \alpha_i = 0,\ \alpha_{i-1} \circ \alpha_i = 0,\ \beta_i \circ \beta_{i-1} = 0\}_{1 \leq i \leq n},$$

where the first group of relations are the clockwise length two cycles around the vertices i, the second group are the length two paths from i to $i-2$ and the latter group are the length two paths from $i-2$ to i, where $1 \leq i \leq n$.

Indecomposable Objects

We propose the following conjecture regarding the number of indecomposable objects in ${}^m\mathbf{Perv}(\mathbb{P}^n)$.

Conjecture 6.3.2.1. *In* ${}^m\mathbf{Perv}(\mathbb{P}^n)$ *there are* $\displaystyle\sum_{i=1}^{n+1} i^2$ *(isomorphism classes of) indecomposable objects.*

The idea is to use induction on the number of strata to prove Conjecture 6.3.2.1. One can add the open stratum S_n to $\mathbb{P}^{n-1}$ (or equivalently the closed stratum S_0), and use the inductive hypothesis to know that in ${}^m\mathbf{Perv}(\mathbb{P}^{n-1})$ there are $\sum_{i=1}^{n} i^2$ indecomposable objects. One should then show that there are $(n+1)^2$ objects which are extensions over the closed stratum S_0. One can use (a stronger inductive hypothesis and assume) that there are n pairwise non isomorphic objects in ${}^m\mathbf{Perv}(\mathbb{P}^{n-1})$ with no quotient on S_0 (and dually n pairwise non isomorphic objects with no sub-object on S_0). The intermediate extension of these objects give rise to n pairwise non isomorphic objects in ${}^m\mathbf{Perv}(\mathbb{P}^n)$ with no quotient on S_0 (and dually n pairwise non isomorphic objects with no sub-object on S_0). Extending the longest object in length among the n pairwise non isomorphic objects in ${}^m\mathbf{Perv}(\mathbb{P}^{n-1})$ with no quotient on S_0 under $j_!$ or j_* (depending on the parity of n) gives rise to another indecomposable object in ${}^m\mathbf{Perv}(\mathbb{P}^n)$ with no quotient on S which is not isomorphic to the other n. Dually, the same holds for the longest object in length among the n pairwise non isomorphic objects in ${}^m\mathbf{Perv}(\mathbb{P}^{n-1})$ with no sub-object on S_0. In this way, one gets $n+1$ pairwise non isomorphic objects in ${}^m\mathbf{Perv}(\mathbb{P}^n)$ with no quotient on S_0, and dually $n+1$ pairwise non isomorphic objects in ${}^m\mathbf{Perv}(\mathbb{P}^n)$ with no sub-object on S_0. However, one cannot use Theorem 5.1.4.3 to conclude that there are $(n+1)^2$ object which are extensions over S_0 as the considered objects are not extensions of the same perverse sheaf on ${}^m\mathbf{Perv}(\mathbb{P}^n)$.

Minimal Projective Resolutions and Global Dimension

We now consider minimal projective presentations of simple objects in $\mathcal{S}_i \in {}^m\mathbf{Perv}(\mathbb{P}^n)$, that is exact sequences of the form

$$\mathcal{P}(\ker \pi_i) \to \mathcal{P}_i \overset{\pi_i}{\twoheadrightarrow} \mathcal{S}_i,$$

where $\mathcal{P}_i \in {}^p\mathbf{Perv}(X)$ is the projective cover of $\mathcal{S}_i$ with projective cover map π_i and $\mathcal{P}(\mathcal{E})$ is the projective cover of the object $\mathcal{E} \in {}^p\mathbf{Perv}(X)$. We can generalise (6.3) and note that

$$\ker(\pi_i) \cong \mathrm{rad}(\mathcal{P}_i) \cong \begin{cases} \widehat{\mathcal{P}}_1 & \text{if } i = 0 \\ \mathcal{S}_{i-1} \oplus \widehat{\mathcal{P}}_{i+1} & \text{if } 1 \le i \le n-1 \\ \mathcal{S}_{n-1} & \text{if } i = n \end{cases}, \tag{6.10}$$

where $\widehat{\mathcal{P}}_i$ is the projective cover of the simple object $\mathcal{S}_i$ in ${}^m\mathbf{Perv}(\mathbb{P}^i)$ for $1 \le i \le n-1$ and $\mathcal{P}(\widehat{\mathcal{P}}_i) \cong \mathcal{P}_i \in {}^m\mathbf{Perv}(\mathbb{P}^n)$. Therefore, the minimal projective presentations of simple objects are

$$\begin{aligned}
&\mathcal{P}_1 \to \mathcal{P}_0 \twoheadrightarrow \mathcal{S}_0,\\
&\mathcal{P}_{i-1} \oplus \mathcal{P}_{i+1} \to \mathcal{P}_i \twoheadrightarrow \mathcal{S}_i \quad \text{if } 1 \leq i \leq n-1,\\
&\mathcal{P}_{n-1} \to \mathcal{P}_n \twoheadrightarrow \mathcal{S}_n.
\end{aligned}$$

The minimal projective presentation can be extended inductively to obtain minimal projective resolutions, see Section 3.5.1. We find that

$$\begin{aligned}
&\mathcal{P}_0^\bullet = \mathcal{P}_2 \hookrightarrow \mathcal{P}_1 \to \mathcal{P}_0,\\
&\mathcal{P}_i^\bullet = \mathcal{P}_2 \hookrightarrow \mathcal{P}_1 \to \mathcal{P}_0 \oplus \mathcal{P}_2 \to \ldots \to \mathcal{P}_{i-1} \oplus \mathcal{P}_{i+1} \to \mathcal{P}_i \quad \text{if } 1 \leq i \leq n
\end{aligned} \tag{6.11}$$

Thus, we can conclude that $\mathrm{gldim}({}^m\mathbf{Perv}(\mathbb{P}^n)) = 2n$.

AR-Quiver

We now explain how we expect one could build the Auslander-Reiten quiver of ${}^m\mathbf{Perv}(\mathbb{P}^n)$. We proceed by induction, by adding the open stratum S_n to $\mathbb{P}^{n-1}$. By using the argument after Conjecture 6.3.2.1, we can build the $(n+1)^2$ indecomposable extensions with support containing S_n.

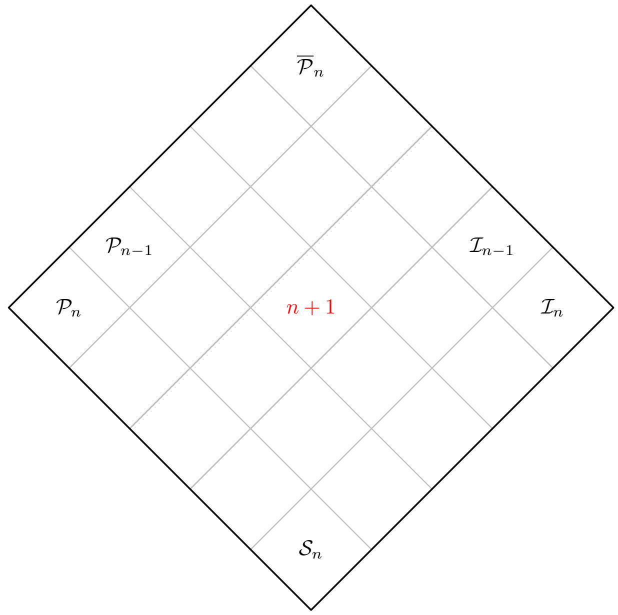

Figure 6.9: Perverse Sheaves in ${}^m\mathbf{Perv}(\mathbb{P}^n)$ whose support contains S_n.

In a similar way to Section 6.2.2, one should check that all the morphisms between objects in Figure 6.9, which can go either up and right or down and right, are irreducible. Furthermore, one needs to check that the irreducible morphisms which appear in Figure 6.9 are all the irreducible morphisms.

Moreover, by inductive hypothesis, we can consider understood perverse sheaves on ${}^m\mathbf{Perv}(\mathbb{P}^{n-1})$, in particular the Auslander-Reiten quiver of it. Therefore, it remains to check how we connect new square to the existing Auslander-Reiten quiver $\mathrm{Q}_m(\mathbb{P}^{n-1})$.

Since $\mathrm{rad}(\mathcal{P}_n) \cong \mathcal{S}_{n-1}$, the morphism $\mathcal{S}_{n-1} \hookrightarrow \mathcal{P}_n$ is irreducible. Moreover any indecomposable object $\mathcal{B} \in {}^m\mathbf{Perv}(\mathbb{P}^n)$ such that

$$\begin{tikzcd} \mathcal{P}_n \arrow[rr, two heads] \arrow[dr] & & \mathcal{S}_n \\ & \mathcal{B} \arrow[ur] & \end{tikzcd},$$

that is any object sitting on the bottom left edge of the $(n+1)^2$ square of Figure 6.9, is such that $\mathcal{B} \cong P_!(\mathcal{E})$ for some $\mathcal{E} \in {}^m\mathbf{Perv}(S_n)$. Moreover, there are short exact sequences

$$0 \to \mathcal{A} \xrightarrow{\theta} \mathcal{B} \to \mathcal{S}_n \to 0$$

in ${}^m\mathbf{Perv}(X)$, where $\mathcal{A} \in {}^m\mathbf{Perv}(X)$ is such that there is a factorisation

$$\begin{tikzcd}
\mathcal{S}_{n-1} \arrow[rr, hook] \arrow[dr] & & \widehat{\mathcal{I}}_{n-1} \\
& \mathcal{A} \arrow[ur] &
\end{tikzcd}.$$

Equivalently, the objects $\mathcal{A}$ sit in the top right edge of the n^2 square. In order to check if the morphism $\theta : \mathcal{A} \hookrightarrow \mathcal{B}$ is irreducible, one can apply Lemma 2.2.3.7. For any $\tau : \mathcal{E} \to \mathcal{S}_n$, we need to check that either $\exists \tau_1 : \mathcal{E} \to \mathcal{B}$ such that $\tau = \beta \circ \tau_1$ or that $\exists \tau_2 : \mathcal{B} \to \mathcal{E}$ such that $\beta = \tau \circ \tau_2$, that is

$$\begin{tikzcd}
& & & \mathcal{E} \arrow[dl, shift right, "\tau_2"'] \arrow[d, "\tau"] & \\
0 \arrow[r] & \mathcal{A} \arrow[r, "\theta"] & \mathcal{B} \arrow[r, "\theta'"] \arrow[ur, shift right, "\tau_1"'] & \mathcal{S}_n \arrow[r] & 0.
\end{tikzcd}$$

Note that τ cannot be a monomorphism as $\mathcal{S}_n \in {}^m\mathbf{Perv}(\mathbb{P}^n)$ is a simple object. Moreover, since $j^*\mathcal{E} \neq 0$, the support of $\mathcal{E}$ contains S_n, that is it appears in the $(n+1)^2$ square. Moreover, since $\mathcal{E}$ maps into $\mathcal{S}_n$ we have that $\mathcal{E} \cong P_!(\mathcal{F})$ for some $\mathcal{F} \in {}^m\mathbf{Perv}(U)$. Therefore, depending on the length of $\mathcal{E}$ the maps $\tau_1 : \mathcal{E} \to \mathcal{B}$ and $\tau_2 : \mathcal{B} \to \mathcal{E}$ exist. This explains how to connect the top right edge of the (n^2) square to the bottom left edge of the $(n+1)^2$ square and dually how to connect the bottom right edge of the $(n+1)^2$ square to the top left edge of the (n^2).

Since $\mathrm{rad}(\mathcal{P}_{n-1}) \cong \mathcal{P}_n \oplus \mathcal{S}_{n-2}$, the morphism $\mathcal{S}_{n-2} \hookrightarrow \mathcal{P}_{n-1}$ is irreducible. The above considerations for the short exact sequences

$$0 \to \mathcal{C} \to \mathcal{D} \to \overline{\mathcal{P}}_n \to 0$$

where $\mathcal{C} \cong i_* P_!(\mathcal{F})$ and $\mathcal{D}$ sits in the factorisation

$$\begin{tikzcd}
\mathcal{P}_{n-1} \arrow[rr, two heads] \arrow[dr] & & \overline{\mathcal{P}}_n \\
& \mathcal{D} \arrow[ur] &
\end{tikzcd}$$

shows how to connect the bottom right edge of the $(n-1)^2$ square to the top left edge of the $(n+1)^2$ square. Dually, this also show how to attach the bottom left edge of the $(n-1)^2$ square to the top right edge of the $(n+1)^2$ square. Therefore, we expect the

Auslander-Reiten quiver of the category ${}^m\mathbf{Perv}(\mathbb{P}^n)$ to look as follows.

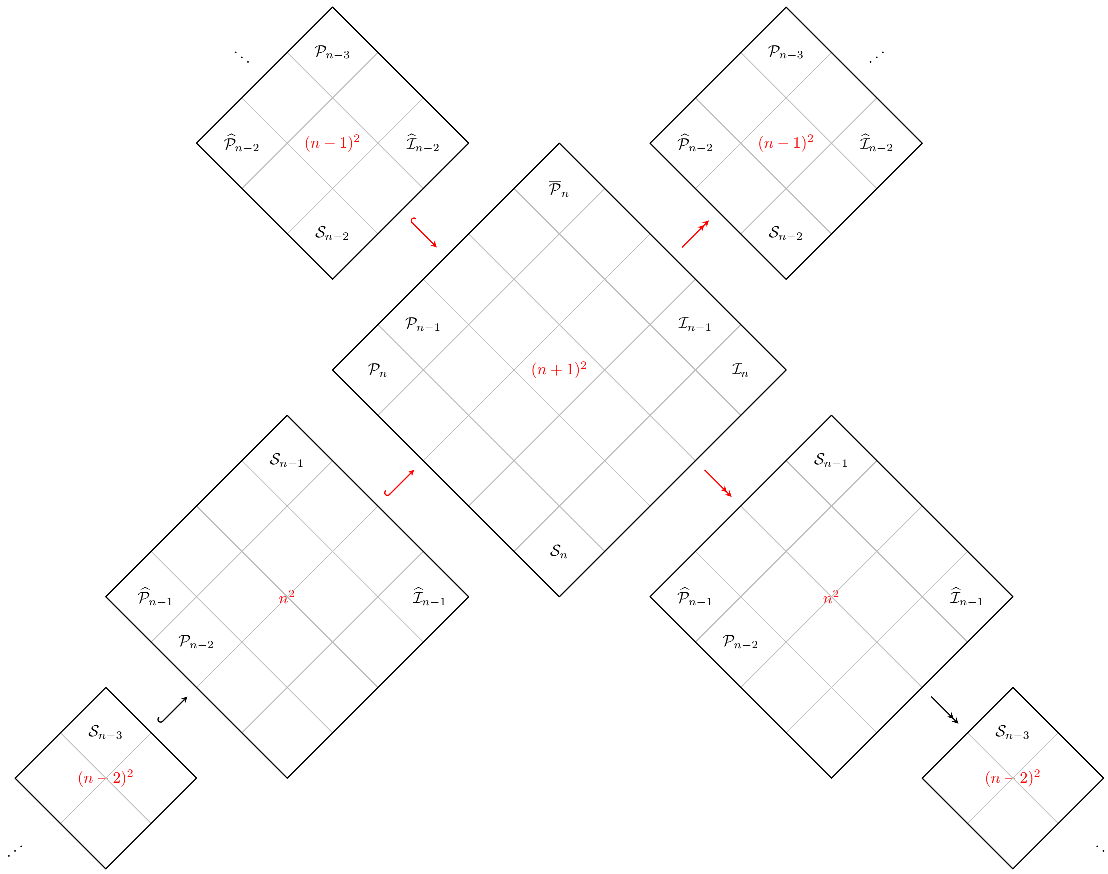

Figure 6.10: Conjectured Auslander-Reiten quiver of ${}^m\mathbf{Perv}(\mathbb{P}^n)$.

Note that in Figure 6.10, all the odd squares, that is the ones of the form $(n-k)^2$ for k odd sit on the diagonal from the bottom left corner to the top right corner. Dually, all the even squares, that is the ones of the form $(n-k)^2$ for k even sit on the diagonal from the bottom right corner to the top left corner. The arrows in Figure 6.10, that is irreducible morphisms between indecomposable objects, go either up and right or down and right. Moreover, all the i^2 squares appear twice for $0 \leq i \leq n$, while the $(n+1)^2$ square appears only once in the middle. The objects with support containing S_0 are placed in the two diagonals of Figure 6.10. In particular, the two diagonals are given by the factorisations in irreducible morphisms of the projective cover map $\mathcal{P}_0 \twoheadrightarrow \mathcal{S}_0$ and injective hull map $\mathcal{S}_0 \hookrightarrow \mathcal{I}_0$. The two diagonals intersect in the object $\mathcal{L}_n$, which denotes the longest object in ${}^{m}\mathbf{Perv}(\mathbb{P}^n)$ (that is the unique object with dimension vector $(1, 2, 2, \ldots, 2)$).

We are able to compute the Auslander-Reiten quiver for ${}^{m}\mathbf{Perv}(\mathbb{P}^n)$ for $n \leq 5$. The hardest part is to show that one can find all irreducible morphisms.

On Faithfulness

The heart ${}^{m}\mathbf{Perv}(\mathbb{P}^n)$ is faithful by [Bei87b].

6.4 Final Remarks, Open Questions and Future Research

In this section, we ask some questions related to the results we presented earlier.

Remark 6.4.0.1. *In Theorem 3.4.0.6, the hypothesis that the characteristic of $\Bbbk$ does not divide the order of the fundamental group $\pi_1(S)$ for any stratum $S \subset X$ can be dropped. In fact, such hypothesis implies that ${}^{p}\mathbf{Perv}(S)$ is semisimple and this is not used at all in the construction of projective covers of simple objects, see Section 3.3.1 and 3.3.2.*

Remark 6.4.0.2. *All the algebras that appear in Chapter 6 are string algebras, see [BR87]. String algebras have a combinatorial description which can be used to establish Conjecture 6.3.2.1 and to determine the Auslander-Reiten quiver.*

Question 6.4.0.3. *With regards to what was explained in Section 3.5.4, can we give a geometric meaning to the construction of the Auslander-Reiten translations?*

In Remark 3.6.0.11, we noted that for a faithful heart ${}^{p}\mathbf{Perv}(X) \subset \mathbf{D}_{\mathrm{c}}(X)$ we can give a bound for the global dimension. This implies that there exists a Serre functor on $\mathbf{D}_{\mathrm{c}}(X)$.

Question 6.4.0.4. *In such situation, can we explicitly describe the Serre functor?*

In Remark 5.1.2.7, we noted that in general the converse of Lemma 5.1.2.5 does not hold.

Question 6.4.0.5. *Does the converse of Lemma 5.1.2.5 hold for small extensions?*

In Example 5.1.4.6, we give an example of quiver with relations that cannot appear as perverse sheaves on a topologically stratified space.

Question 6.4.0.6. *Which quivers with relations can be realised as ${}^{p}\mathbf{Perv}(X)$ for a topologically stratified space X.*

For $X = \mathbb{P}^2$, the only faithful heart is given by ${}^m\mathbf{Perv}(\mathbb{P}^2)$, see Section 6.2.

Question 6.4.0.7. *Is ${}^m\mathbf{Perv}(\mathbb{P}^n)$ the only faithful heart in $\mathbf{D}_c(\mathbb{P}^n)$?*

A positive answer would mean that the result in [Bei87b] on faithfulness is an if and only if.

In Example 4.1.2.6 we gave an instance of an infinite representation type algebra.

Question 6.4.0.8. *Can we characterise finite and infinite representation type in terms of conditions on the links?*

In Section 6.2.4, we showed that ${}^s\mathbf{Perv}(\mathbb{P}^2)$ is an example of a non-quadratic algebra.

Question 6.4.0.9. *Is it possible to generalise the example of ${}^s\mathbf{Perv}(\mathbb{P}^2)$ to $\mathbb{P}^n$?*

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